MATH 250, Spring 2017

Final Exam

1. (25 points) Compute each of the following.

(a) The two solutions to the equation $12x + 9y = 15$ with the smallest positive x-values.

(b) The sum of the divisors of 3000.

(c) The smallest positive integer that leaves a remainder of 2 when divided by 12, a remainder of -1 when divided by 7 and is divisible by 5.

(d) A solution to $x^{47} \equiv 11 \pmod{100}$.

(e) Consider the elliptic curve $y^2 = x^3 + 1$ and the point $P = (2, 3)$. Find the smallest positive value of n such that $nP = \mathcal{O}$. Find $8P$. (Recall, $\mathcal{O}$ is the point at infinity and $nP = \underbrace{P + P + \cdots + P}_{n \text{ times}}$).

2. (40 points) For each of the following, either give a short proof of the statement <u>or</u> give a counter example.

(a) A number is divisible by 2^n if and only if its last n digits are divisible by 2^n.

(b) $n^3 + 1$ is composite for every natural number $n \geq 2$.

(c) If $a = bq + r$ then $\gcd(a, q) = \gcd(q, r)$.

(d) If $a = bq + r$ then $\gcd(a, b) = \gcd(q, r)$.

(e) The number $(a - b)(a - c)(b - c)$ is even for all integers a, b, c.

(f) The equation $x^2 + 3y^4 = 7z^2$ has no nonzero integer solutions.

(g) The equation $(x + y)^2 = 79 + 2xy$ has no integer solutions.

3. (15 points) Let $p \geq 5$ be prime. Show that $x^2 - 3y^2 \equiv 0 \pmod{p}$ has nontrivial solutions if and only if p is congruent to 1 or 11 modulo 12.

4. (20 points) [Exercise 43.2 from textbook]

(a) Find all solutions to the Diophantine equation $y^2 = x^5+1$ modulo 7. How many solutions are there?

(b) Find all solutions to the Diophantine equation $y^2 = x^5 + 1$ modulo 11. How many solutions are there?

(c) Let p be a prime with the property that $p \not\equiv 1 \pmod{5}$. Prove that the Diophantine Equation $y^2 = x^5 + 1$ has exactly p solutions modulo p.