Math 455 Topology, Spring 2017
Final Exam
May 8

1. (20 points) For the following problems, just write T or F.
   (a) (2 points) There is an isotopy from the left-handed trefoil knot to the unknot.
   (b) (2 points) There is a homeomorphism from $\mathbb{R}^3$ to $\mathbb{R}^3$ which maps a left-handed trefoil knot to a right-handed trefoil knot.
   (c) (2 points) S^{126} doesn't admit a continuous nowhere vanishing vector field.
   (d) (2 points) The Euler characteristic of $\partial\Delta^4$ is 2.
   (e) (2 points) Let A, B, C be closed surfaces. If $A \ncong B$, then $A\#C \ncong B\#C$.
   (f) (2 points) Let A, B, C be spaces. If $A \ncong B$, then $A \times C \ncong B \times C$.
   (g) (2 points) A space consisting of finitely many points is compact in any topology.
   (h) (2 points) A path-connected space is always connected.
   (i) (2 points) The product of two connected spaces is always connected.
   (j) (2 points) There is a retraction from S^1 to the point $(1, 0) \in S^1$.

2. (8 points) Prove this version of **the pasting lemma**: Let A and B be open subsets of the space X and $A \cup B = X$. If $f : A \to Y$ and $g : B \to Y$ are continuous functions and they agree over $A \cap B$, namely $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h : X \to Y$ defined by $h(x) := f(x)$ if $x \in A$ and $h(x) := g(x)$ if $x \in B$ is also a continuous function.

3. (12 points)
   (a) (6 points) Let X be a Hausdorff space and A a compact subset of X. Prove that A is closed in X.
   (b) (6 points) Let X be a Hausdorff space and Y its one-point compactification. Prove that the original topology $\mathcal{T}$ on X and the subspace topology $\mathcal{T}'$ which X inherites from Y are the same.

4. (10 points) Prove that $\mathbb{R}P^1$, defined as the quotient space obtained from S^1 by identifying each pair of antipodal points, is homeomorphic to S^1.

5. (8 points) Let $\alpha : I \to X$ and $\beta : I \to X$ be two paths in the space $X = \mathbb{R}^2 \backslash \{(0, 0)\}$ defined by
$$\alpha(s) = (\cos(\pi s), \sin(\pi s)) \text{ and } \beta(s) = (\cos(\pi s), -\sin(\pi s)).$$
Prove that with the end points fixed, α cannot be deformed continuously to β in X. More precisely, show that $\alpha \not\simeq \beta$ rel $\{0, 1\}$. Justify all your claims.

6. (7 points) Prove that $\mathbb{R}^m \cong \mathbb{R}^n$ if and only if $m = n$.

7. (10 points) Let X be the path-connected and compact triangulable space $\mathbb{R}P^4$. Its homology groups are as follows.

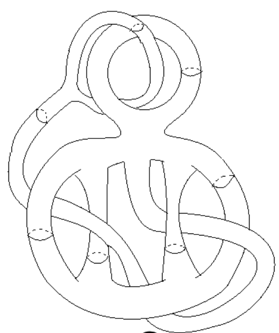
$$H_0(X) \cong \mathbb{Z}, H_1(X) \cong \mathbb{Z}/2\mathbb{Z}, H_2(X) \cong 0, H_3(X) \cong \mathbb{Z}/2\mathbb{Z}, H_i(X) \cong 0, i \geq 4.$$

(a) (2 points) Compute the Euler characteristic $\chi(X)$.

(b) (4 points) Prove that any map $f : X \to X$ has a fixed-point.

(c) (4 points) Can X be a topological group? (Clearly state any theorem you use and prove any statement you make.)

8. (15 points) Two closed surfaces S_1 and S_2 are shown below.



S_1

S_2

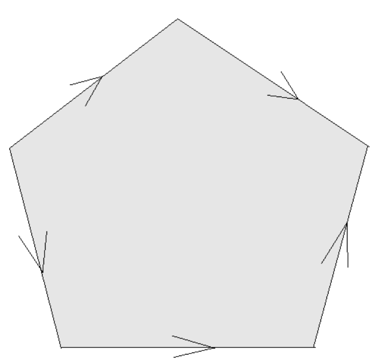
(a) (3 points) Identify each of the two surfaces with a standard surface on the list of the classification theorem.

(b) (3 points) Sketch their polygonal models.

(c) (3 points) Compute their fundamental groups.

(d) (6 points) Let C_1 be a compact surface obtained from S_1 by removing the interiors of 3 disjoint closed discs. Let C_2 be a compact surface obtained from S_2 by removing the interiors of 5 disjoint closed discs. Compute all the homology groups of C_1 and C_2.

9. (10 points) X is the space obtained by identifying all the five edges of a solid pentagon (area is filled in) along directions shown below.



(a) (5 points) Use the Seifert-van Kampen Theorem to compute the fundamental group of X.

(b) (5 points) Is X is contractible? Justify your claim.