Math 450, Spring 2018

Final Exam, Due Tuesday, May 8, at 4pm SHARP!!!

Instructions: Do all the problems. **You must fully justify your answers.** You may use theorems (and lemmas, etc.) from class or the book to do so, but please say enough about any theorem you quote (its name, if it has one, number if it's an unnamed theorem in the book, or a **brief** description for a theorem from class) that I know what you're talking about. You may also use results from prerequisite classes and from assigned homework problems. However, unless otherwise noted, you may *not* quote theorems we didn't cover, unassigned exercises, or challenge problems; for example, you may not use Theorem 12.64, or Exercise 15 of Chapter 8, unless you prove it from scratch. If you are not sure whether some argument or statement requires further justification, please ask me about it.

You may use the book (Wheeden & Zygmund) and **your own notes**. You **may not use** other books, notes other than your own, online information, other people, or any other outside sources. You **may not discuss the problems** with anyone other than me. (Not even to mention to your grandmother that, say, "Number 14 is pretty easy.") On the other hand, please feel free to come talk to me about the exam. I won't necessarily give very helpful responses (depending on what you ask), but you never know.

The exam is due at **4pm** on the **Tuesday** of exam period. You may hand it to me in person in my office, or slip it under my door in an envelope. No extensions will be granted; any exam not handed in by 4:00 **sharp** will be graded as a zero.

There are six problems, worth a total of 120 points, plus an optional bonus problem worth 3 points.

---

1. **(10 points)** Let $E \subseteq \mathbb{R}^n$ be measurable, let f be a measurable function on E, and let $\{f_k\}_{k\geq 1}$ be a sequence of measurable functions on E. Prove that $f_k \xrightarrow{m} f$ on E if and only if for every $\varepsilon > 0$, there exists $N \geq 1$ such that

$$\mu\big(\{x \in E : |f_n(x) - f(x)| > \varepsilon\}\big) < \varepsilon \quad \text{for all } n \geq N.$$

2. **(15 points)** Let $f \in L^1(\mathbb{R}^n)$. For every $\varepsilon > 0$, prove that there is a measurable set $E \subseteq \mathbb{R}^n$ with $\mu(E) < \infty$ such that $\left| \int_E f\, d\mu - \int_{\mathbb{R}^n} f\, d\mu \right| < \varepsilon$.

3. **(20 points)** Let $1 \leq p < q \leq \infty$, and let $E \subseteq \mathbb{R}^n$ be measurable. Let $X = L^p(E) \cap L^q(E)$, and for each $f \in X$, define

$$\|f\|_X = \|f\|_p + \|f\|_q.$$

Prove that $\|\cdot\|_X$ is a norm on X, and in fact that X is a Banach space with respect to the norm $\|\cdot\|_X$.

(over)

4. **(15 points)** Let $f, g : (0,1) \to \mathbb{R}$ be continuous decreasing functions. Suppose that for all $\alpha \in \mathbb{R}$, we have $\mu(\{f > \alpha\}) = \mu(\{g > \alpha\})$. Prove that $f = g$.

(*Hint: Suppose* $f(x_0) < g(x_0)$ *for some* $0 < x_0 < 1$. *What can you say about the sets* $\{f > f(x_0)\}$ *and* $\{g > f(x_0)\}$*?*)

5. **(30 points.)** Let $1 \leq p < \infty$, and let $E \subseteq \mathbb{R}^n$ be measurable. Let $\{f_k\}_{k\geq 1} \subseteq L^p(E)$ and $\{g_k\}_{k\geq 1} \subseteq L^\infty(E)$ be sequences, and suppose that $f_k \to f$ in $L^p(E)$, $g_k \to g$ pointwise, and $\|g_k\|_\infty \leq M$ for some constant $0 < M < \infty$ independent of k. Prove that $f_k g_k \to fg$ in $L^p(E)$.

6. **(30 points.)** Let $\{r_k\}_{k\geq 1}$ be an enumeration of the rational numbers (i.e., each $r \in \mathbb{Q}$ shows up as exactly one r_k), and let $\{c_k\}_{k\geq 1}$ be a sequence of *positive* real numbers such that $\sum_{k\geq 1} c_k$ is finite. Define

$$f(x) = \sum_{k\geq 1} c_k g(x - r_k), \quad \text{where} \quad g(x) = \begin{cases} x^{-1/2} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $f \in L^1(\mathbb{R})$, but $f \notin L^2([a,b])$ for any $a < b \in \mathbb{R}$.

OPTIONAL BONUS. **(3 points.)** Let $E_1, E_2 \subseteq \mathbb{R}^1$ be measurable sets with $|E_1|, |E_2| > 0$. Let

$$D = \{x_1 - x_2 : x_1 \in E_1, x_2 \in E_2\}.$$

Use the Vitali Covering Lemma (Lemma 7.17) to prove that D contains an interval.