On Abelianization of Lie Algebroids and Lie Groupoids

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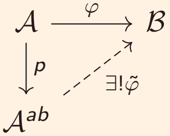

Existence of Abelianization (Algebroids)

Proposition (Contreras & Fernandes)

Let $A \to M$ be a transitive Lie algebroid. Then its abelianization is $A^{ab} = A/[\mathfrak{g}_M, \mathfrak{g}_M]$.

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A bundle of Lie algebras $\mathfrak{g}_M$ has an abelianization if and only if $\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$ is a subbundle. In this case $\mathfrak{g}_M^{ab} = \mathfrak{g}/\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$.

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Proposition

A regular Lie algebroid has an abelianization if its isotropy bundle has an abelianization. In this case $A^{ab} = A/\overline{[\mathfrak{g}_M, \mathfrak{g}_M]}$

Abelianization of Groupoids

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The abelianization of $\mathcal{G} \rightrightarrows M$ *is:*

- $\mathcal{G}^{ab} \rightrightarrows M$ *abelian*
- $p : \mathcal{G} \to \mathcal{G}^{ab}$ *surjective morphism over* Id_M

such that for any $\mathcal{H} \rightrightarrows N$ *abelian and morphism* $\psi : \mathcal{G} \to \mathcal{H}$*, we have:*

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\psi} & \mathcal{H} \\ \downarrow p & \nearrow \exists! \tilde{\psi} & \\ \mathcal{G}^{ab} & & \end{array} .$$

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This definition depends on the category.

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- A Lie groupoid might not have a smooth abelianization:

$$\mathcal{G} = SO(3) \times \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$$

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$$\gamma(i,\epsilon) = x_i, \qquad a(t,i) = a_i(t), \qquad b(i,\epsilon) = 0, \qquad i = 0,1$$

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$$\begin{array}{ccc} T(\Sigma) & \xrightarrow{\quad h \quad} & A \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{\quad \gamma \quad} & M \end{array}$$

$$h|_U = a(t,\epsilon)dt + b(t,\epsilon)d\epsilon$$

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Relations between $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}_g(\mathcal{A})$

- $\mathcal{A}$-homology is courser than $\mathcal{A}$-homotopy:

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- $\mathcal{G}(\mathcal{A})$ smooth $\not\Rightarrow \mathcal{G}_g(\mathcal{A})$ smooth.

Integrating Lie Algebroids

Theorem (Crainic & Fernandes)

For a Lie algebroid $\mathcal{A}$, the following statements are equivalent:

- (a) $\mathcal{A}$ *is integrable.*
- (b) $\mathcal{G}(\mathcal{A})$ *is smooth.*
- (c) *the monodromy groups* $\mathcal{N}_x(\mathcal{A})$ *are locally uniformly discrete.*

Moreover, in this case, $\mathcal{G}(\mathcal{A})$ *is the unique s-simply connected Lie groupoid integrating* $\mathcal{A}$*.*

Ordinary Monodromy

$\mathcal{N}_x$ is the image of:

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when $\Omega(X, Y) := \sigma([X, Y]) - [\sigma(X), \sigma(Y)]$ is $Z(\mathfrak{g}_L)$-valued:

$$\partial_x([\gamma]) = \exp\left(\int_\gamma \Omega\right)$$

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$\mathcal{N}_x^{ext}$ is the image of:

$$\partial_x^{ext} : H_2(\tilde{L}^h, \mathbb{Z}) \to \mathcal{G}(\mathfrak{g}_x^{ab})$$
$$[\gamma] \mapsto \exp\left(\int_\gamma q^*\Omega^{ab}\right).$$

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- $q : \tilde{L}^h \to L$: holonomy cover of L relative to
$$\nabla_X s = [\sigma^{ab}(X), s] \text{ on } \mathfrak{g}_L^{ab}.$$
- $\Omega^{ab} := \sigma^{ab}([X, Y]) - [\sigma^{ab}(X), \sigma^{ab}(Y)]$

Theorem (Contreras & Fernandes)

The extended and ordinary monodromy homomorphisms of a Lie algebroid fit into a commutative diagram:

$$\begin{array}{ccc} \pi_2(L,x) & \xrightarrow{\partial_x} & \mathcal{G}(\mathfrak{g}_x) \\ \downarrow{h_2} & & \downarrow \\ H_2(\tilde{L}^h,\mathbb{Z}) & \xrightarrow{\partial_x^{ext}} & \mathcal{G}(\mathfrak{g}_x^{ab}) \end{array},$$

where h_2 is the Hurewicz map.

Results for Genus Integration

Theorem (Contreras & Fernandes)

Let $\mathcal{A} \to M$ be a transitive Lie algebroid with trivial holonomy. The following statements are equivalent:

- (a) *the extended monodromy groups are discrete;*
- (b) *the genus integraton $\mathcal{G}_g(\mathcal{A})$ is smooth;*
- (c) *the abelianization $\mathcal{A}^{ab}$ has an abelian integration.*

If any of these hold then $\mathcal{G}_g(\mathcal{A})$ has Lie algebroid isomorphic to $\mathcal{A}^{ab}$.

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$$[(X, f), (Y, g)] = ([X, Y], [f, g]_{\mathfrak{g}_M} + \nabla_X g - \nabla_Y f + \Omega[X, Y])$$

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  - $\odot_{X,Y,Z}(\Omega([X, Y], Z) + \nabla_X(\Omega(Y, Z))) = 0$

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In this case, we can compute $\mathcal{N}_x(\mathcal{A})$ and $\mathcal{N}_x(\mathcal{A})$ as follows:

$$\mathcal{N}_x(\mathcal{A}) = \{\exp\left(\int_\gamma \omega\right) : \gamma \in \pi_2(M,x)\},$$

$$\mathcal{N}_x^{ext}(\mathcal{A}) = \{\exp\left(\int_\gamma \omega\right) : \gamma \in H_2(M,\mathbb{Z})\}.$$

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Thank you!