Coisotropcity of fixed points under torus action on the variety of Lagrangian subalgebras

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Introduction

In this talk, I will introduce my recent work on the topic of coisotropic subalgebras of Lie bialgebras, a problem initiated in the work of Marco Zambon. My work is grounded in the theory of semisimple Lie algebras and algebro-geometric methods including linear algebraic groups and toric varieties.
Standard Lie bialgebra structure on $\mathfrak{g}$

- Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, with adjoint algebraic group $G$.
- Let $H$ be a fixed maximal torus of $G$, and $B$ be a fixed Borel subgroup of $G$ containing $H$ with the set of simple roots $\Gamma$. Let $N := [B, B]$ be the unipotent radical of $B$. Denote by $\mathfrak{h}, \mathfrak{b}, \mathfrak{n}$ the Lie algebras of $H, B, N$, respectively.
- Recall that $\mathfrak{g}$ has a standard Lie bialgebra structure induced by the standard Manin triple $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}_{\Delta}, \mathfrak{g}_{st}^*)$, where $\mathfrak{g}_{\Delta} := \{(x, x)|x \in \mathfrak{g}\}$ and $\mathfrak{g}_{st}^* := \mathfrak{h}_{-\Delta} + \mathfrak{n} \oplus \mathfrak{n}^- = \{(x + y, -x + z)|x \in \mathfrak{h}, y \in \mathfrak{n}, z \in \mathfrak{n}^-\}$, and the nondegenerate invariant bilinear form on $\mathfrak{g} \oplus \mathfrak{g}$ is given by

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \ll x_1, y_1 \gg - \ll x_2, y_2 \gg \quad (1)$$

- We regard $\mathfrak{g}$ as $\mathfrak{g}_{\Delta}$, and $\mathfrak{g}^*$ as $\mathfrak{g}_{st}^*$, then $\mathfrak{g} \cong \mathfrak{g}_{\Delta}$ is a Lie bialgebra and $G$ is a Poisson-Lie group.
Define the variety of Lagrangian subalgebras

- A Lie subalgebra $\mathfrak{l}$ of $\mathfrak{g} \oplus \mathfrak{g}$ is called Lagrangian, if $\dim(\mathfrak{l}) = \dim(\mathfrak{g})$ and $\mathfrak{l}$ an isotropic subspace of $(\mathfrak{g} \oplus \mathfrak{g}, \langle \cdot, \cdot \rangle)$.
- Let $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ denote the set of Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$, i.e.
  \[ \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g}) := \{ \mathfrak{l} \subset \mathfrak{g} \oplus \mathfrak{g} | \mathfrak{l} \text{ is a Lagrangian subalgebra of } \mathfrak{g} \oplus \mathfrak{g} \}, \]
- Since being subalgebras and being isotropic are polynomial conditions in $\text{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$, $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ is an algebraic subset in $\text{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$. We call $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ the variety of Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$.
- $G \times G$ acts on $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ via the natural adjoint action.
Examples of Lagrangian subalgebras

- Recall the bilinear form on $\mathfrak{g} \oplus \mathfrak{g}$:

  \[
  \langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle
  \]

- Basic examples: $\mathfrak{g}_\Delta$, $\mathfrak{g}_{-\Delta}$, $\mathfrak{g}_{st}^*$ are Lagrangian subalgebras.

- Other examples:
  - Let $S \subseteq \Gamma$ be a subset of simple roots. Define
    \[
    \varsigma_S := n_S \oplus n_{-S} + (m_S)_{\Delta} \in \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g}),
    \]
    Then $\varsigma_S$ is a Lagrangian subalgebra.
  - Any $(G \times G)$-translation of the aboves are Lagrangian subalgebras.
Define coisotropic subalgebras (Zambon)

Since $\mathfrak{g}$ is a Lie bialgebra, its dual $\mathfrak{g}^*$ is a Lie algebra.

- A subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ is called coisotropic if $\mathfrak{m}^0$ is a subalgebra of $\mathfrak{g}^*$, where

$$\mathfrak{m}^0 := \{ \xi \in \mathfrak{g}^* | \xi(x) = 0, \forall x \in \mathfrak{m} \}$$

is the annihilator of $\mathfrak{m}$ in $\mathfrak{g}^*$.

- (Result from M. Zambon) Coisotropic subalgebras can be embedded into the variety of Lagrangian subalgebras.

  - Recall the identification $\mathfrak{g}^* \cong \mathfrak{g}_{st}^* \subset \mathfrak{g} \oplus \mathfrak{g}$. Let $\mathfrak{m}^\perp$ be the counterpart of $\mathfrak{m}^0$ in $\mathfrak{g}_{st}^* \subset \mathfrak{g} \oplus \mathfrak{g}$. Then it's easy to check that $(\mathfrak{m})_\Delta \oplus \mathfrak{m}^\perp$ is a Lagrangian subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$.

We call a Lagrangian subalgebra $\mathfrak{l}$ coisotropic, if $\mathfrak{l}$ comes from the above construction.
Define the subvariety of coisotropic subalgebras

\[ \mathcal{CL}(g \oplus g) := \{ l \in \mathcal{L}(g \oplus g) \mid l \text{ is a coisotropic Lagrangian subalgebra} \} . \]

- In other words, \( l \in \mathcal{CL}(g \oplus g) \) iff \( l = m_\Delta \oplus m_\perp \) for a coisotropic subalgebra \( m \) of \( g \).
- Fact: \( \mathcal{CL}(g \oplus g) \subset \mathcal{L}(g \oplus g) \) is a subvariety.
- Fact: There is a Poisson structure \( \pi_\mathcal{L} \) on \( \mathcal{L}(g \oplus g) \). If \( l \in \mathcal{CL}(g \oplus g) \) then \( \pi_\mathcal{L}(l) = 0 \).
- Fact: \( H_\Delta \subset G \times G \) acts on \( \mathcal{CL}(g \oplus g) \).
Strategy

It’s hard to determine coisotropic subalgebras of \( \mathfrak{g} \oplus \mathfrak{g} \). Instead, I study the \( H_\Delta \)-fixed points on \( \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g}) \).

- Structure of \( \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g}) \) (irreducible components, \( (G \times G) \)-orbits, Poisson structure, etc.) is well studied. (Sam Evens and Jiang-Hua Lu, papers on Lagrangian subalgebras)

- Can study \( H_\Delta \)-fixed points in \( \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g}) \) at first, then check the coisotropicty of these fixed points.

- \( \overline{G} := (G \times G) \cdot g_\Delta \subset \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g}) \) is one of the irreducible componets of \( \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g}) \), which is called the wonderful compactification of \( G \) (De Concini-Procesi).

- In this talk, I will focus on \( \overline{G} \), give its \( H_\Delta \) fixed points and check their coisotropicty.

- \( (\overline{G})^{H_\Delta} \) turns out to be a union of toric varieties. This fixed point set is describable in terms of combinatorics of Weyl group.
Irreducible components of fixed point set

- $\zeta_0 := h_\Delta + n \oplus n^- \in \mathcal{L}(g \oplus g)$
- Let $\overline{G}^{H_\Delta} = \bigcup X_i$ be the irreducible decomposition. Since $\overline{G}^{H_\Delta}$ is smooth (a result by Iversen, 1972), the irreducible components $X_i$’s are connected components.
- Fact: each irreducible component $X_i$ must contain some point of the form $(y, w) \cdot \zeta_0$. Define $X_{y,w}$ to be the irreducible component containing $(y, w) \cdot \zeta_0$. Therefore, each irreducible component $X_i$ must be of the form $X_{y,w}$
- It suffices to study $X_{y,w}$’s, which turn out to be toric varieties.
More notations

- For a subset $S \subseteq \Gamma$, the standard parabolic subalgebra $p_S$ has the Levi decomposition $p_S = m_S \oplus n_S$ and the opposite $H$-stable nilradical $n_\bar{S}$. Let $g_S := [m_S, m_S]$ with the corresponding adjoint group $G_S$.

- Let $\zeta_S := n_S \oplus n_\bar{S} + (m_S)_\Delta \in \mathcal{L}(g \oplus g)$ and $K_S := (H \times H) \cdot \zeta_S \subset \mathcal{L}(g \oplus g)$. Then I show that $K_S$ is a toric variety for the torus $H_S := G_S \cap H$.

- For any $y, w \in W$, define $I_{y,w} := \{ \alpha \in \Gamma | y \cdot \alpha = w \cdot \alpha \}$

- For $w \in W$, let $\Phi_w := \{ \gamma \in \Phi^+ | w^{-1}(\gamma) \in \Phi^- \}$. 


Recall: toric variety

- Definition: A toric variety $X$ is a normal variety that contains a torus $T$ as a dense open subset, together with an action map $T \times X \to X$ which extends the natural action of $T$ on itself.

- (Roughly) A fan in $\mathbb{R}^n$ is a collection of "cones".

- Fact: A toric variety $X$ is completely determined by its fan $\text{Fan}(X)$. The cones in the fan $\text{Fan}(X)$ give an affine open cover for $X$.

- Examples of toric varieties: projective spaces
Main results (part I)

- The subvariety $\Sigma_S := \bigsqcup_{J \subseteq S} (H \times H) \cdot \zeta_J$ is isomorphic to $\mathbb{C}^{\left| S \right|}$, and it is an open affine subset in $K_S$. In particular, the dimension of $K_S$ is $\left| S \right|$.
- Let $H_S$ be the maximal torus of $G_S$. Then $K_S$ is a toric variety for the torus $H_S$, with an open affine cover

$$K_S = \bigcup_{\nu \in W_S} (\nu, \nu) \Sigma_S$$

Its fan is given by the Weyl chamber decomposition of $\mathfrak{g}_S$, for which the Weyl group is $W_S$.
- $X_{y,w} = (y, w) \cdot K_{I_{y,w}}$ is a (translation of) toric variety.
Main results (part II)

- There exists coisotropic points in $X_{y,w}$ if and only if the following conditions hold:
  1. $(wy^{-1})^2 = 1$
  2. $\Phi_y \cap \Phi_w = \emptyset$

- If the above conditions hold, then there are finitely many coisotropic points in $X_{y,w}$, each one corresponds to a subset $J$ of $I_{y,w}$.

- Thus, the coisotropic points in $\overline{G}^{H_\Delta}$ form a finite set.
We look at the example when $g = \mathfrak{sl}(2, \mathbb{C})$.

- $G = PGL(2, \mathbb{C}) \subset P(M^{2\times 2}(\mathbb{C}))$, take $H$ to be diagonal matrices of $G$.

- Let $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in M^{2\times 2}(\mathbb{C})$

- Fact: $\overline{G} \cong P(M^{2\times 2}(\mathbb{C}))$. Moreover, there exists a $G \times G$-equivariant isomorphism from $P(M^{2\times 2}(\mathbb{C}))$ to $\overline{G} \subset \mathcal{L}(g \oplus g)$, which sends $[id]$ to $g_{\Delta}$.

- The $H_{\Delta}$-fixed points in $P(M^{2\times 2}(\mathbb{C}))$ are
  \begin{align*}
  \{ \text{diagonal matrices} \} \sqcup \{ [E] \} \sqcup \{ [F] \}.
  \end{align*}

- In the language of Lagrangian subalgebras, the $H_{\Delta}$-fixed points in $\overline{G}$ are $(H \times H) \cdot g_{\Delta}$, $(e, s) \cdot \zeta_0$ and $(s, e) \cdot \zeta_0$.

- In the above, $\zeta_0 := \mathfrak{h}_{\Delta} + \mathfrak{n} \oplus \mathfrak{n}^-$, and $s \in W$ is the order 2 element in the Weyl group of $\mathfrak{sl}(2)$. The three pieces of the disjoint union are toric varieties for $H$, $\{1\}$ and $\{1\}$ respectively.
On the other irreducible components

The above results show that the $H_\Delta$-fixed points in $\bar{G}$ are union of toric varieties, in which the coisotropic locus are discrete.
I can apply the same method to the other irreducible components of $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$:

- For a general irreducible component $L(S, T, d)$ of $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$, its $H_\Delta$-fixed points are products of toric varieties and a homogeneous space of a special orthogonal group. Among these fixed points, the coisotropic points along the toric variety factors are discrete.
Thank you!