Coisotropicity of fixed points under torus action on the variety of Lagrangian subalgebras

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Outline

Introduction

In this talk, I will introduce my recent work on the topic of coisotropic subalgebras of Lie bialgebras, a problem initiated in the work of Marco Zambon. My work is grounded in the theory of semisimple Lie algebras and algebro-geometric methods including linear algebraic groups and toric varieties.

Standard Lie bialgebra structure on $\mathfrak{g}$

- Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, with adjoint algebraic group G.
- Let H be a fixed maximal torus of G, and B be a fixed Borel subgroup of G containing H with the set of simple roots Γ. Let $N := [B, B]$ be the unipotent radical of B. Denote by $\mathfrak{h}, \mathfrak{b}, \mathfrak{n}$ the Lie algebras of H, B, N, respectively.
- Recall that $\mathfrak{g}$ has a standard Lie bialgebra structure induced by the standard Manin triple $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}_\Delta, \mathfrak{g}^*_{st})$, where $\mathfrak{g}_\Delta := \{(x, x) | x \in \mathfrak{g}\}$ and $\mathfrak{g}^*_{st} := \mathfrak{h}_{-\Delta} + \mathfrak{n} \oplus \mathfrak{n}^- = \{(x + y, -x + z) | x \in \mathfrak{h}, y \in \mathfrak{n}, z \in \mathfrak{n}^-\}$, and the nondegenerate invariant bilinear form on $\mathfrak{g} \oplus \mathfrak{g}$ is given by

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \ll x_1, y_1 \gg - \ll x_2, y_2 \gg \tag{1}$$

- We regard $\mathfrak{g}$ as $\mathfrak{g}_\Delta$, and $\mathfrak{g}^*$ as $\mathfrak{g}^*_{st}$, then $\mathfrak{g} \cong \mathfrak{g}_\Delta$ is a Lie bialgebra and G is a Poisson-Lie group.

Define the variety of Lagrangian subalgebras

- A Lie subalgebra $\mathfrak{l}$ of $\mathfrak{g} \oplus \mathfrak{g}$ is called Lagrangian, if $dim(\mathfrak{l}) = dim(\mathfrak{g})$ and $\mathfrak{l}$ an isotropic subspace of $(\mathfrak{g} \oplus \mathfrak{g}, \langle\,,\,\rangle)$.
- Let $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ denote the set of Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$, i.e.

$$\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g}) := \{\mathfrak{l} \subset \mathfrak{g} \oplus \mathfrak{g} | \mathfrak{l} \textit{ is a Lagrangian subalgebra of } \mathfrak{g} \oplus \mathfrak{g}\},$$

- Since being subalgebras and being isotropic are polynomial conditions in $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$, $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ is an algebraic subset in $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$. We call $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ the variety of Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$.
- $G \times G$ acts on $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ via the natural adjoint action.

Examples of Lagrangian subalgebras

- Recall the bilinear form on $\mathfrak{g} \oplus \mathfrak{g}$:

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \ll x_1, y_1 \gg - \ll x_2, y_2 \gg \tag{2}$$

- Basic examples: $\mathfrak{g}_\Delta$, $\mathfrak{g}_{-\Delta}$, $\mathfrak{g}^*_{st}$ are Lagrangian subalgebras.
- Other examples:
  - Let $S \subseteq \Gamma$ be a subset of simple roots. Define

$$\zeta_S := \mathfrak{n}_S \oplus \mathfrak{n}_S^- + (\mathfrak{m}_S)_\Delta \in \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g}),$$

    Then ζ_S is a Lagrangian subalgebra.
- Any $(G \times G)$-translation of the aboves are Lagrangian subalgebras.

Define coisotropic subalgebras (Zambon)

Since $\mathfrak{g}$ is a Lie bialgebra, its dual $\mathfrak{g}^*$ is a Lie algebra.

- A subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ is called coisotropic if $\mathfrak{m}^0$ is a subalgebra of $\mathfrak{g}^*$, where
$$\mathfrak{m}^0 := \{\xi \in \mathfrak{g}^* | \xi(x) = 0, \forall x \in \mathfrak{m}\}$$
is the annihilator of $\mathfrak{m}$ in $\mathfrak{g}^*$.
- (Result from M. Zambon) Coisotropic subalgebras can be embedded into the variety of Lagrangian subalgebras.
  - Recall the identification $\mathfrak{g}^* \cong \mathfrak{g}^*_{st} \subset \mathfrak{g} \oplus \mathfrak{g}$. Let $\mathfrak{m}^\perp$ be the counterpart of $\mathfrak{m}^0$ in $\mathfrak{g}^*_{st} \subset \mathfrak{g} \oplus \mathfrak{g}$. Then it's easy to check that $(\mathfrak{m})_\Delta \oplus \mathfrak{m}^\perp$ is a Lagrangian subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$.

  We call a Lagrangian subalgebra $\mathfrak{l}$ coisotropic, if $\mathfrak{l}$ comes from the above construction.

Define the subvariety of coisotropic subalgebras

$$\mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g}) := \{\mathfrak{l} \in \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g}) | \mathfrak{l} \textit{ is a coisotropic Lagrangian subalgebra}\}.$$

- In other words, $\mathfrak{l} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$ iff $\mathfrak{l} = \mathfrak{m}_\Delta \oplus \mathfrak{m}^\perp$ for a coisotropic subalgebra $\mathfrak{m}$ of $\mathfrak{g}$.
- Fact: $\mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g}) \subset \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ is a subvariety.
- Fact: There is a Poisson structure $\pi_{\mathcal{L}}$ on $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$. If $\mathfrak{l} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$ then $\pi_{\mathcal{L}}(\mathfrak{l}) = 0$.
- Fact: $H_\Delta \subset G \times G$ acts on $\mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$.

Strategy

It's hard to determine coisotropic subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$. Instead, I study the H_Δ-fixed points on $\mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$.

- Structure of $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ (irreducible components, $(G \times G)$-orbits, Poisson structure, etc.) is well studied. (Sam Evens and Jiang-Hua Lu, papers on Lagrangian subalgebras)
- Can study H_Δ-fixed points in $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ at first, then check the coisotropicity of these fixed points.
- $\overline{G} := \overline{(G \times G) \cdot \mathfrak{g}_\Delta} \subset \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ is one of the irreducible componets of $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$, which is called the wonderful compactification of G (De Concini-Procesi).
- In this talk, I will focus on $\overline{G}$, give its H_Δ fixed points and check their coisotropicity.
- $(\overline{G})^{H_\Delta}$ turns out to be a union of toric varieties. This fixed point set is describable in terms of combinatorics of Weyl group.

Irreducible components of fixed point set

- $\zeta_0 := \mathfrak{h}_\Delta + \mathfrak{n} \oplus \mathfrak{n}^- \in \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$
- Let $\overline{G}^{H_\Delta} = \cup X_i$ be the irreducible decomposition. Since $\overline{G}^{H_\Delta}$ is smooth (a result by Iversen, 1972), the irreducible components X_i's are connected components.
- Fact: each irreducible component X_i must contain some point of the form $(y, w) \cdot \zeta_0$. Define $X_{y,w}$ to be the irreducible component containing $(y, w) \cdot \zeta_0$. Therefore, each irreducible component X_i must be of the form $X_{y,w}$
- It suffices to study $X_{y,w}$'s, which turn out to be toric varieties.

More notations

- For a subset $S \subseteq \Gamma$, the standard parabolic subalgebra $\mathfrak{p}_S$ has the Levi decomposition $\mathfrak{p}_S = \mathfrak{m}_S \oplus \mathfrak{n}_S$ and the opposite H-stable nilradical $\mathfrak{n}_S^-$. Let $\mathfrak{g}_S := [\mathfrak{m}_S, \mathfrak{m}_S]$ with the corresponding adjoint group G_S.
- Let $\zeta_S := \mathfrak{n}_S \oplus \mathfrak{n}_S^- + (\mathfrak{m}_S)_\Delta \in \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ and $K_S := \overline{(H \times H) \cdot \zeta_S} \subset \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$. Then I show that K_S is a toric variety for the torus $H_S := G_S \cap H$.
- For any $y, w \in W$, define $I_{y,w} := \{\alpha \in \Gamma | y \cdot \alpha = w \cdot \alpha\}$
- For $w \in W$, let $\Phi_w := \{\gamma \in \Phi^+ | w^{-1}(\gamma) \in \Phi^-\}$.

Recall: toric variety

- Definition: A toric variety X is a normal variety that contains a torus T as a dense open subset, together with an action map $T \times X \longrightarrow X$ which extends the natural action of T on itself.
- (Roughly) A fan in $\mathbb{R}^n$ is a collection of "cones".
- Fact: A toric variety X is completely determined by its fan $Fan(X)$. The cones in the fan $Fan(X)$ give an affine open cover for X.
- Examples of toric varieties: projective spaces

Main results (part I)

- The subvariety $\Sigma_S := \sqcup_{J \subseteq S}(H \times H) \cdot \zeta_J$ is isomorphic to $\mathbb{C}^{|S|}$, and it is an open affine subset in K_S. In particular, the dimension of K_S is $|S|$.
- Let H_S be the maximal torus of G_S. Then K_S is a toric variety for the torus H_S, with an open affine cover

$$K_S = \bigcup_{v \in W_S} (v, v)\Sigma_S$$

  Its fan is given by the Weyl chamber decomposition of $\mathfrak{g}_S$, for which the Weyl group is W_S.
- $X_{y,w} = (y, w) \cdot K_{I_{y,w}}$ is a (translation of) toric variety.

Main results (part II)

- There exists coisotropic points in $X_{y,w}$ if and only if the following conditions hold:
  (1) $(wy^{-1})^2 = 1$ (2) $\Phi_y \cap \Phi_w = \emptyset$
- If the above conditions hold, then there are finitely many coisotropic points in $X_{y,w}$, each one corresponds to a subset J of $I_{y,w}$.
- Thus, the coisotropic points in $\overline{G}^{H_\Delta}$ form a finite set.

Rank 1 example

We look at the example when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$.

- $G = PGL(2, \mathbb{C}) \subset P(M^{2\times 2}(\mathbb{C}))$, take H to be diagonal matrices of G.
- $$Let\ E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in M^{2\times 2}(\mathbb{C})$$
- Fact: $\overline{G} \cong P(M^{2\times 2}(\mathbb{C}))$. Moreover, there exists a $G \times G$-equivariant isomorphism from $P(M^{2\times 2}(\mathbb{C}))$ to $\overline{G} \subset \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$, which sends $[id]$ to $\mathfrak{g}_\Delta$.
- The H_Δ-fixed points in $P(M^{2\times 2}(\mathbb{C}))$ are $\{diagonal\ \ matrices\} \sqcup \{[E]\} \sqcup \{[F]\}$.
- In the language of Lagrangian subalgebras, the H_Δ-fixed points in $\overline{G}$ are $\overline{(H \times H) \cdot \mathfrak{g}_\Delta}$, $(e, s) \cdot \zeta_0$ and $(s, e) \cdot \zeta_0$.
- In the above, $\zeta_0 := \mathfrak{h}_\Delta + \mathfrak{n} \oplus \mathfrak{n}^-$, and $s \in W$ is the order 2 element in the Weyl group of $\mathfrak{sl}(2)$. The three pieces of the disjoint union are toric varieties for H, $\{1\}$ and $\{1\}$ respectively.

On the other irreducible components

The above results show that the H_Δ-fixed points in $\overline{G}$ are union of toric varieties, in which the coisotropic locus are discrete.
I can apply the same method to the other irreducible components of $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$:

- For a general irreducible component $L(S, T, d)$ of $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$, its H_Δ-fixed points are products of toric varieties and a homogeneous space of a special orthogonal group. Among these fixed points, the coisotropic points along the toric variety factors are discrete.

Thank you!