Geometry of the Kepler Problem and the Kepler-Lorentz Duality

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Abstract

It is known that the dynamical symmetry of the Kepler problem allows one to map the phase space curves of the problem to geodesics on cotangent bundles of 3-manifolds. The scattering state Kepler problem, also known as Rutherford scattering, is symmetric under the action of the Lorentz group $SO(1,3)$, and its phase space curves map to geodesics on the cotangent bundle of 3-hyperboloid H^3. I present new scattering variables in the momentum space that allows the scattering state Kepler problem to be cast in a form that is in an exact correspondence with that of the relativistic free particle. I call this correspondence the Kepler-Lorentz duality. Using these variables, I present the dualities in the boost representations of $SO(1,3)$ and $SL(2,\mathbb{C})$ and in the corresponding four-vectors and spinors. Extending the symmetry, I consider the dual problems in the contexts of conformal symmetry and twistors, where the various energy regimes of the Kepler problem are brought together in a unified context and the equivalence of the dual problems to the 4D inverted harmonic oscillator is shown. In light of the recent connections between the symmetries of the Kepler problem and of planar $\mathcal{N}=4$ super Yang-Mills theory, I discuss a similar possible analogous relationship between Kepler-Lorentz duality and gauge-gravity duality.

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To my mother

Chapter 1

Introduction

Gravity and electromagnetism: these are the two forces of nature that are readily experienced at the scale of a human being. Both are classically described by the same force law, called the **inverse-square law**, where the strength of the force is inversely proportional to the square of the distance from its source. The study of motions of a body subject to an inverse-square force is called the **Kepler problem**, with the Hamiltonian

$$H = \frac{p^2}{2m} - \frac{k}{r}, \tag{1.1}$$

where k is the coupling constant. Due to the presence of an extra symmetry unique to the system, the Kepler problem has many special properties that manifest themselves classically as well as quantum mechanically. For example, the bound states of the Kepler problem and the harmonic oscillator are the only classical central-force systems that have closed orbits for every possible set of initial conditions (Bertrand's theorem). The hydrogen atom, a quantum mechanical bound-state Kepler problem, is fully quantizable using only the algebraic commutation relations. Rutherford (Coulomb) scattering, a quantum mechanical scattering-state Kepler problem, has equal classical and quantum scattering cross-sections. In this thesis, I propose an example where seemingly different systems are revealed to be different manifestations of the same mathematical problem by the consideration of their symmetries.

In Chapter 2, we present the dynamical symmetry of the Kepler problem, where the problem is revealed to possess an extra symmetry upon the consideration of its momentum space. We consider two compelling roads, one algebraic and the other geometric, to quantization. In quantization, we follow closely Ch. 2 of [1] and Ch. 4 of [2]. This chapter captures our motivation for investigating the symmetries of the Kepler problem.

In Chapter 3, we show two methods of regularization, where we place the Kepler problem in more natural settings in which infinities do not occur. The Moser method and the Kustaanheimo-Stiefel method not only help in regularization, but also reveal deep insights about the symmetries of the Kepler problem. Here, we again follow closely

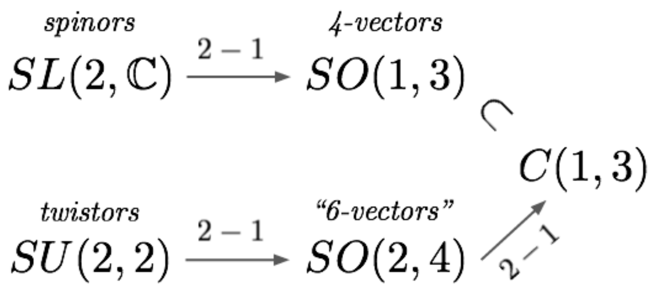

FIGURE 1.1: Symmetry groups, their relationships, and objects on which they act

the treatment of the Moser method in [1] and the of the Kustaanheimo-Stiefel method in [2]. This chapter provides some of the essential mathematics that is required for Chapter 5. In fact, we will revisit the Moser method and the Kustaanheimo-Stiefel method again.

In Chapter 4, we cast, motivated by the shared symmetry group $SO(1,3)$, the scattering state Kepler problem in a form that is in an exact correspondence with the relativistic free particle. We call this correspondence the Kepler-Lorentz duality. We present the duality by showing the explicit forms of the boost transformations of $SO(1,3)$ and the four-vectors corresponding to 4-velocity, 4-momentum, and 4-position in the respective dual contexts. The boost transformations of $SL(2,\mathbb{C})$, a double cover of $SO(1,3)$, is also naturally obtained through the duality, and we present a possible dual extension of the corresponding quantity in the spinor formulation of relativity. Apart from some background material, the ideas presented in this chapter are wholly original to the author.

In Chapter 5, we lift the symmetry of the Kepler-Lorentz duality to the conformal group $C(1,3)$ of Minkowski space $\mathbb{R}^{1,3}$, where various energy regimes of the Kepler problem are brought together in a unified context as an extension of the Moser method[1]. We also recast the problem in the context of twistor theory, where we show the equivalence of the 3d bound/scattering Kepler problem to the 4d simple/inverted harmonic oscillator by revisiting the Kustaanheimo-Stiefel method. Though a majority of the mathematical results presented in the chapter is grounded on Ch. 6-7 of Ref. [2], we provide considerably different physical interpretations of the results using our knowledge of the Kepler-Lorentz duality. We also simplify the presentation considerably by providing a more intuitive interpretation of the dense mathematics. Remarkably, the conformal symmetry gives an extended phase manifold of the Kepler problem, where some group-geometric and dimensional properties of the Kepler-Lorentz duality seem to be related to that of gauge-gravity duality, originally formulated in the context of string theory. In light of the recent connections between the symmetries of the Kepler problem and (planar) $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, we suggest that the

[1]The groups $SO(2,4)$ and $SU(2,2)$ are respectively double and quadruple covers of the conformal group $C(1,3)$ of Minkowski space.

AdS_5/CFT_4 duality may be a high energy extension of the classical symmetries apparent in the Kepler-Lorentz duality. In addition, the Kustaanheimo-Stiefel map, naturally obtained via twistor variables, suggests the equivalence of the 3D scattering state Kepler problem, 4D relativistic free-particle kinematics, and 4D inverted harmonic oscillator. Both seem to be exciting future research directions.

In our presentation, we assume the reader has a working knowledge of the basics of Lie groups and algebras, differential geometry, and Hamiltonian mechanics. Figure 1.1 summarizes the layout of this thesis.

Chapter 2

A Higher Symmetry

The Kepler problem exhibits a greater symmetry than is expected in the general class of central force problems. Namely, the Kepler problem is symmetric under generalized rotations in $\mathbb{R}^4$ or $\mathbb{R}^{1,3}$ rather than simply in $\mathbb{R}^3$. I will first motivate the greater symmetry of the problem by introducing an extra conserved quantity, the Laplace-Runge-Lenz vector. Then, I will associate the group of symmetries of the Kepler problem with corresponding invariant geometries by providing two related methods of quantization. A solid understanding of the interplay between group theory and geometry will be essential to our subsequent discussions.

2.1 Dynamical Symmetry

In a central force problem in 3-dimensional Euclidean space, we expect the energy H and the angular momentum $\mathbf{L}$ to be conserved by time-translational and 3D rotational symmetry, respectively. Such problems with fixed H generally remain invariant under the group action of $SO(3)$, the group of proper rotations in $\mathbb{R}^3$. In the special case of Kepler potentials ($\sim 1/r$), however, there is an extra conserved quantity called the **Laplace-Runge-Lenz (LRL) vector $\mathbf{A}$**:

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\hat{\mathbf{r}}, \tag{2.1}$$

where k is the coupling constant. The LRL vector $\mathbf{A}$ always lies in the plane of motion and points toward the perihelion, the point of closest approach in the orbit. Noether's theorem tells us that there is a conserved current for every continuous symmetry, and vice versa. Thus, we should suspect that Kepler problems possess a higher group of symmetries. Given this vector, I will motivate the presence of an extra symmetry by two independent but related lines of reasoning, one algebraic and the other geometric. The discussion in this section is kept at a heuristic level and is meant to give an overview of the topics that will be explored further.

2.1.1 Algebraic Motivation

The quantities H, $\mathbf{L}$, and $\mathbf{A}$ contribute a total of 7 conserved scalars (vector quantities each contribute 3 scalar components). However, two relations, $\mathbf{A}\cdot\mathbf{L} = 0$ and $A^2 = m^2k^2 + 2mHL^2$, impose two constraints and yields 5 independent conserved quantities. A d-dimensional Hamiltonian system possesses a maximum of $2d-1$ integrals of motion. Since the Kepler problem possesses the maximum number of integrals of motion, we call it **maximally superintegrable** [3]. Motion for a maximally superintegrable classical Hamiltonian system follows a 1-dimensional closed curve with respect to time evolution in phase space. The quantum mechanical analogues of such systems are also fully quantized using just commutation relations, as will be shown in Section 2.2.

Given such quantities, the algebraic structure of the Kepler problem can be made explicit through the use of Poisson brackets. The **Poisson bracket** is defined as

$$\{f, g\} = \sum_{i=1}^{N}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$$

with canonical coordinates (q_i, p_i) on phase space and functions $f(q_i, p_i, t)$ and $g(q_i, p_i, t)$ [4]. The Poisson bracket operation, in its essence, yields the infinitesimal transformation of $f(q_i, p_i, t)$ with respect to parameter s associated with $g(q_i, p_i, t)$, and vice versa. Thus, f and g are called **infinitesimal generators** of the Lie algebra. As we will see, Poisson brackets provide a nice bridge between classical and quantum mechanics through its interpretation as quantum mechanical commutators under canonical quantization.

Let us first write the angular momentum and LRL vector in component form:

$$L_k = \epsilon_{ijk} r_i p_j, \qquad \bar{A}_k = \frac{A_k}{p_0} = \frac{1}{p_0}\left(\epsilon_{ijk} p_i L_j - mk\frac{r_k}{r}\right), \tag{2.2}$$

where $\bar{A}_k$ is the LRL vector rescaled by $p_0 = \sqrt{2m|H|}$ and where I have used the Einstein summation convention. I will continue to use this convention unless stated otherwise. With H held fixed, their Poisson bracket relations yield the following [5]:

$$\{L_i, L_j\} = \epsilon_{ijk} L_k, \qquad \{\bar{A}_i, L_j\} = \epsilon_{ijk}\bar{A}_k, \qquad \{\bar{A}_i, \bar{A}_j\} = +\epsilon_{ijk} L_k \tag{2.3}$$

for $H < 0$ (bound state) and

$$\{L_i, L_j\} = \epsilon_{ijk} L_k, \qquad \{\bar{A}_i, L_j\} = \epsilon_{ijk}\bar{A}_k, \qquad \{\bar{A}_i, \bar{A}_j\} = -\epsilon_{ijk} L_k \tag{2.4}$$

for $H > 0$ (scattering state). Thus, the components of $\mathbf{L}$ and $\bar{\mathbf{A}}$ are closed under the Poisson bracket operation. We note that Eq. (2.3), with a positive sign in the last relation, forms a Lie algebra isomorphic to that of $SO(4)$, the group of rotations in $\mathbb{R}^4$. Also, Eq. (2.4), with a negative sign in the last relation, forms a Lie algebra isomorphic to that of the Lorentz group $SO(1,3)$, which includes both 3-rotations and 3-boosts in $\mathbb{R}^{1,3}$.[1] Thus, the $H<0$ (bound state) problem is symmetric under the group action of $SO(4)$ and the $H>0$ (scattering state) problem is symmetric under the group action of $SO(1,3)$. We will not discuss the $H=0$ case, where the rescaling of the LRL vector with p_0 becomes problematic. It is also worthwhile to look at the algebra of the linear combinations of $\mathbf{L}$ and $\bar{\mathbf{A}}$ [5]:

$$\mathbf{M} = \frac{1}{2}(\mathbf{L} + \bar{\mathbf{A}}), \qquad \mathbf{N} = \frac{1}{2}(\mathbf{L} - \bar{\mathbf{A}}),$$

$$\{M_i, M_j\} = \epsilon_{ijk} M_k, \qquad \{N_i, N_j\} = \epsilon_{ijk} N_k, \qquad \{M_i, N_j\} = 0 \tag{2.5}$$

The components of $\mathbf{M}$ and $\mathbf{N}$ are independently closed under the Poisson bracket operation, and we verify that the components taken together form a Lie algebra isomorphic to that of $SO(3)\times SO(3) \simeq SO(4)$. We can also find the Casimir elements, $M^2 = M_1^2 + M_2^2 + M_3^2$ and $N^2 = N_1^2 + N_2^2 + N_3^2$:

$$M^2 = N^2 = -\frac{1}{4}\frac{mk^2}{2H}. \tag{2.6}$$

The quantum analogues of $\mathbf{M}$ and $\mathbf{N}$ will be useful in the algebraic method of quantization presented in Section 1.2. As expected, taking the Poisson brackets of L_i, $\bar{A}_i$, M_i, and N_i with H give zeros:

$$\{L_i, H\} = 0, \qquad \{\bar{A}_i, H\} = 0, \qquad \{M_i, H\} = 0, \qquad \{N_i, H\} = 0.$$

2.1.2 Geometric Motivation

Now that we have found the higher symmetry groups for the Kepler problem, we ask, what is the natural geometry on which the Kepler problem takes place that remains invariant under the action of the symmetry groups? Indeed, the invariance of the problem under the higher dimensional symmetry groups is not immediately obvious. The clue to identifying such a geometric surface comes from considering the *momentum space*—rather than the familiar position space—of the problem. To do this, we use a cute trick involving the LRL vector $\mathbf{A}$ [6]. First, we rearrange Eq. (2.1):

$$\mathbf{p} \times \mathbf{L} - \mathbf{A} = mk\hat{\mathbf{r}}.$$

Taking the dot product with itself and even-permuting the triple product, we get

$$p^2L^2 + A^2 - \mathbf{L}\cdot(\mathbf{A}\times\mathbf{p}) = m^2k^2.$$

[1]For connectedness, we take the conditions proper for $SO(4)$ and proper and orthochronous for $SO(1,3)$ to be implicit throughout our discussion.

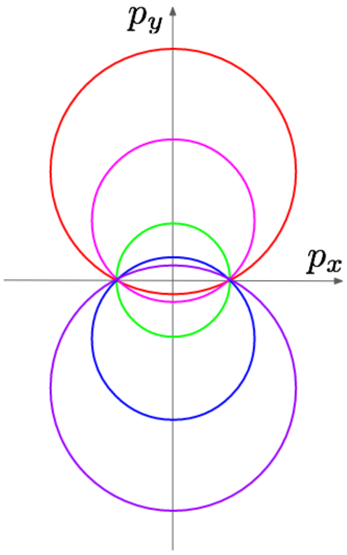

FIGURE 2.1: Momentum space curves of the bound Kepler problem

The angular momentum **L** is conserved and we are free to pick a plane on which the motion is constrained. We pick **A** to lie on the x-axis and **L** to lie on the z-axis:

$$(p_x^2 + p_y^2)L^2 + A^2 - 2ALp_y = m^2k^2.$$

Dividing through by L^2 and factoring, we have

$$p_x^2 + \left(p_y - \frac{A}{L}\right)^2 = \left(\frac{mk}{L}\right)^2, \tag{2.7}$$

an equation for a circle with $radius = mk/L$ centered at $(p_x, p_y) = (0, A/L)$. Since all of the coupling constant k's that appear in Eq. (2.7) are squared, the momentum space curves are circles in both $k < 0$ (attractive) and $k > 0$ (repulsive) Kepler problems. Therefore, all particle paths in the position space of the Kepler problem, which we know to be conic sections, trace out circles (or parts of circles) in momentum space! A series of equal energy curves with varying L for various signs of H and k is shown in Fig. 2.1.

This is neat, but what is its connection to geometry? To see the connection, let us first consider the $H < 0$ (bound state) Kepler problem. In this problem, all equal energy curves in momentum space share two common points on the p_x-axis, as seen in Fig. 2.1. Given the radius and the center of the circle on the p_y-axis, we can find the distance d from the origin to one of the common points on the p_x-axis using the Pythagorean theorem:

$$d^2 = \left(\frac{mk}{L}\right)^2 - \left(\frac{A}{L}\right)^2 = -2mH,$$

where we used $A^2 = m^2k^2 + 2mHL^2$. Notice that d is just $p_0 = \sqrt{2m|H|}$ defined in the previous subsection for $H < 0$. So all circles at fixed H share two common points at

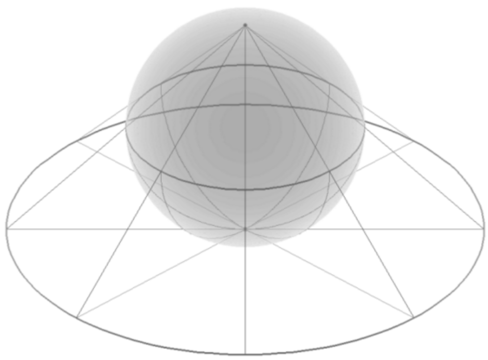

FIGURE 2.2: Stereographic projection

$p_x = \pm p_0$.

We recall the notion of stereographic projection typically encountered in a first course in complex analysis, where one compactifies the complex plane to a Riemann sphere by mapping points on the plane to the sphere via a line through the North Pole (Fig. 2.2). The stereographic projection is a smooth, conformal (angle-preserving), and one-to-one map that maps circles on the plane to circles on the sphere, and vice versa. Here, we perform this projection in real space. If we choose the radius of the sphere S^2 to be p_0, all of the circular paths in the momentum plane $\mathbb{R}^2$ with fixed energy $p_0 = \sqrt{2m|H|}$ will be mapped to circles in S^2 that contain the antipodal points $p_x = \pm p_0$ on the sphere. Since circles on S^2 that touch the antipodes must be great circles, all particle paths on the momentum plane $\mathbb{R}^2$ will be mapped to great circles, which are **geodesics** on S^2 [7]. In our derivation of Eq. (2.7), we arbitrarily chose a momentum plane $\mathbb{R}^2$ on which the motion is constrained. In fact, we are allowed to consider the more general case of the 3D momentum space $\mathbb{R}^3$ and map this to the 3-sphere S^3, also on which the geodesics are great circles. Switching to normalized dimensionless projective coordinates (X_0, X_1, X_2, X_3), S^3 can be expressed

$$X_0^2 + X_1^2 + X_2^2 + X_3^2 = 1. \tag{2.8}$$

With this generalization, we have a manifold on which $SO(4)$ symmetry of the Kepler problem acts in a natural manner. In other words, 4-rotations of S^3 generate equal energy curves in the momentum space of the bound state Kepler problem.

A similar procedure can be carried out for the $H > 0$ (scattering state) Kepler problem. The natural manifold in this regime turns out to be the 3-hyperboloid H^3, which, in normalized dimensionless coordinates (X_0, X_1, X_2, X_3) as above, can be expressed

$$X_0^2 - X_1^2 - X_2^2 - X_3^2 = 1. \tag{2.9}$$

Recalling that H^3 is also the invariant manifold under Lorentz transformations in special relativity, we verify that the natural manifold under the group action of $SO(1,3)$ in the $H > 0$ Kepler problem is indeed H^3. Thus, 3-rotations and 3-boosts of H^3 generate equal energy curves in the momentum space of the scattering state Kepler problem.

All fine points aside, the remarkable insight gained through the geometric perspective is the following: *the extra symmetry in the Kepler problem allows us to find a more natural manifold on which the motion takes place, and the symmetric transformations of such a manifold allow us to generate all equal energy curves in momentum space.* Since this extra symmetry is only fully evident upon consideration of the dynamics—the momentum space—of the problem, we say that the Kepler problem possesses a **dynamical symmetry**. The classical geometric perspective illustrated in this section will be made more rigorous and explicit in Chapter 2.

It turns out that the identification of a natural manifold for the Kepler problem is even more powerful in connection to quantum mechanics. The manifold allows for a global quantization scheme of the Kepler problem, which will be worked out in Section 1.3. But first, we show that, as a maximally superintegrable system, the energy eigenvalues of the hydrogen atom can be obtained simply by considering the algebra of the problem.

2.2 Pauli Quantization

Algebraic approaches often provide a more elegant and insightful view into unearthing the essence of physical systems than do their analytic counterparts. For example, the algebraic method for quantum harmonic oscillators involves a clever construction of ladder operators that yield the equally spaced energy levels, whereas the analytic method is generally applicable, but feels rather brute force and provides little insight. One can also take an algebraic approach to canonically quantize the hydrogen atom, a $H < 0$ (bound state) Kepler problem. Pauli used this approach to calculate the hydrogen atom energy levels in his 1926 paper, a year before the discovery of the Schrodinger equation [1, 2, 8].

To find a quantum mechanical analogue of the algebraic relations of the classical $H < 0$ Kepler problem, we must first learn about canonical quantization [2, 9]. Canonical quantization is motivated by the remarkable similarity between the Poisson bracket form of the classical Hamilton's equations of motion and the equations of motion for operators in the Heisenberg picture, where the operators, rather than the states, exhibit dynamics. A function $f(q_i, p_i)$ on phase space without explicit time-dependence satisfies

$$\frac{df}{dt} = \{f, H\} \tag{2.10}$$

in Hamiltonian mechanics, while an operator $\hat{f}(\hat{q}_i, \hat{p}_i)$ acting on the Hilbert space satisfies

$$\frac{d\hat{f}}{dt} = \frac{1}{i}[\hat{f}, \hat{H}] \tag{2.11}$$

in the Heisenberg picture, if we define the commutator $[\cdot, \cdot]$ of two operators $\hat{f}$ and $\hat{g}$ as $[\hat{f}, \hat{g}] = \hat{f}\hat{g} - \hat{g}\hat{f}$. The similarities in Eq. (2.10) and Eq. (2.11) suggest the following: For every classical observable $f(q_i, p_i)$ on phase space of a canonical Hamiltonian system, we can associate a linear Hermitian operator $\hat{f}(\hat{q}_i, \hat{p}_i)$ acting on the Hilbert space. Such a map $\mathcal{Q} : f(q_i, p_i) \to \hat{f}(\hat{q}_i, \hat{p}_i)$ is called **quantization**, and $\mathcal{Q}$ preserves the canonical structure. In canonical quantization, we typically choose

$$q_i \xrightarrow{\mathcal{Q}} \hat{q}_i = q_i, \qquad p_i \xrightarrow{\mathcal{Q}} \hat{p}_i = \frac{1}{i}\frac{\partial}{\partial q_i}.$$

After quantization $\mathcal{Q}$ of functions into operators, we still see that Eq. (2.10) differs from Eq. (2.11) by the Poisson bracket. Noting the correspondence

$$\{\cdot, \cdot\} \leftrightarrow \frac{1}{i}[\cdot, \cdot] \tag{2.12}$$

between the Poisson bracket and the commutator, we canonically quantize Eq. (2.10) into Eq. (2.11). Although canonical quantization only works in general for algebra of the polynomials that are quadratic under the Poisson bracket, the Lie algebra of the hydrogen atom meets this criteria.

Let us now canonically quantize the classical infinitesimal generators of $so(4)$, the Lie algebra of the hydrogen atom, which we have written in Eq. (2.2) [1, 2]. Quantization $\mathcal{Q}$ of the angular momentum components L_k is simply

$$L_k \xrightarrow{\mathcal{Q}} \hat{L}_k = \epsilon_{ijk} r_i \hat{p}_j. \tag{2.13}$$

We need to, however, symmetrize the LRL operators $\hat{A}_k$ obtained from the quantization $\mathcal{Q}$ of rescaled LRL vector components $\bar{A}_k$ in order to make them Hermitian:

$$\bar{A}_k \xrightarrow{\mathcal{Q}} \hat{A}_k = \sqrt{\frac{\hat{H}^{-1}}{2m}} \left[\frac{1}{2}\epsilon_{ijk}(\hat{p}_i \hat{L}_j - \hat{L}_i \hat{p}_j) - mk\frac{r_k}{r} \right], \tag{2.14}$$

where $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{k}{r}$ is the Hamiltonian operator. To avoid excessive notation, we will take $\hat{A}_k$ as the quantized Hermitian operator of the rescaled LRL components $\bar{A}_k$ rather than of the regular LRL components A_k. Following the correspondence in Eq. (2.12), we obtain the commutation relations of the operators from Poisson bracket relations of the observables in Eq. (2.3):

$$[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{L}_k, \qquad [\hat{A}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{A}_k, \qquad [\hat{A}_i, \hat{A}_j] = i\epsilon_{ijk}\hat{L}_k. \tag{2.15}$$

A direct calculation also leads to Eq. (2.15), but it is rather tedious. As in the classical case, $\hat{L}_i$ and $\hat{A}_i$ generate the Lie algebra of $SO(4)$. Likewise, Eq. (2.5) becomes

$$\hat{M}_i = \frac{1}{2}(\hat{L}_i + \hat{A}_i), \qquad \hat{N}_i = \frac{1}{2}(\hat{L}_i - \hat{A}_i),$$

$$[\hat{M}_i, \hat{M}_j] = i\epsilon_{ijk}\hat{M}_k, \qquad [\hat{N}_i, \hat{N}_j] = i\epsilon_{ijk}\hat{N}_k, \qquad [\hat{M}_i, \hat{N}_j] = 0. \tag{2.16}$$

Also as before, $\hat{M}_i$ and $\hat{N}_i$ together generate the Lie algebra of $SO(3) \times SO(3) \simeq SO(4)$. The Casimir operators, $\hat{M}^2 = \hat{M}_1^2 + \hat{M}_2^2 + \hat{M}_3^2$ and $\hat{N}^2 = \hat{N}_1^2 + \hat{N}_2^2 + \hat{N}_3^2$, give

$$\hat{M}^2 = \hat{N}^2 = -\frac{1}{4}\left(\frac{mk^2}{2}\hat{H}^{-1} + I\right), \tag{2.17}$$

where I is the identity. Eq. (2.17) differs from the classical Casimir element in Eq. (2.6) by just $-\frac{1}{4}$, and this difference comes from symmetrizing the LRL operator $\hat{A}_i$. The commutation of $\hat{L}_i$, $\hat{A}_i$, $\hat{M}_i$, and $\hat{N}_i$ with the Hamiltonian $\hat{H}$ give zeros, as in the classical case:

$$[\hat{L}_i, \hat{H}] = 0, \qquad [\bar{A}_i, \hat{H}] = 0, \qquad [\hat{M}_i, \hat{H}] = 0, \qquad [\hat{N}_i, \hat{H}] = 0.$$

Let us now consider just the Lie algebras and note that $so(3) = su(2)$, the algebra of Pauli matrices, because multiple Lie groups can share the same Lie algebra. Then $\hat{M}_i$ and $\hat{N}_i$ each generates the algebra $su(2)$. Each irreducible representation of $su(2)$ is specified by a half-integer j, where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$. From the study of spins in quantum mechanics, we know that the Casimir operators $\hat{M}^2$ and $\hat{N}^2$, whose components $\hat{M}_i$ and $\hat{N}_i$ each generate $su(2)$, have eigenvalues $j(j+1)$. Thus, for eigenvalue H of $\hat{H}$, it follows that

$$\hat{M}^2 = \hat{N}^2 = j(j+1) = -\frac{1}{4}\left(\frac{mk^2}{2H} + 1\right).$$

Rearranging,

$$H = -\frac{mk^2}{2(2j+1)^2}.$$

Let us define the principal quantum number $n = 2j+1$, which ranges over $n = 1, 2, 3, \cdots$ for $j = 0, \frac{1}{2}, 1, \cdots$. Then, the energy levels of the hydrogen atom is

$$H = -\frac{mk^2}{2n^2} = -\frac{m_e e^4}{2(4\pi\epsilon_0)^2\hbar^2}\frac{1}{n^2}. \tag{2.18}$$

The $\hbar^2$ term was inserted into the last expression to match the form that is familiar to readers. Eq. (2.18) exhibits the characteristic $1/n^2$ dependence.

Now, why doesn't the energy level of hydrogen depend on other quantum numbers, such as l and m? To see why, let us observe what we get when we take the number of m for a given l and sum over the possible values of l for a given n. The number of m for a given l is $2l+1$, and the possible values of l for a given n range is from 0 to $2j$:

$$\sum_{l=0}^{2j}(2l+1) = (2j+1)^2 = n^2.$$

The n^2 factor in the hydrogen energy level simply comes from considering the degeneracies of m and l for a given n! Thus, the symmetries of the hydrogen atom give rise to degeneracies, which in turn results in the n^2 factor in the energy level [10]. This fact is indeed what allows us to calculate the energies purely from algebraic considerations.

As a last note, the reader may be worried about the inverse of Hamiltonian operator $\hat{H}^{-1}$ appearing in Eq. (2.14) and Eq. (2.17). However, it will be evident from our discussion of classical regularization in Chapter 2 that H^{-1}, rather than H, is the more natural quantity to consider from the perspective of geometry and group theory.

2.3 Fock Quantization

The merit of the algebraic method in Pauli Quantization lies in its elegance, but the intuitive picture of the dynamical symmetry in the quantum mechanical scheme remains yet elusive. In the classical scheme, we have identified the natural manifolds S^3 and H^3 on which the momentum space curves of bound and scattering Kepler problems are geodesics. We saw in Pauli Quantization that the quantum mechanical Kepler problem preserves the group of symmetries seen in the classical Kepler problem. Thus, we can map the quantum mechanical problem onto the manifolds S^3 and H^3 to obtain the eigenstates and energy eigenvalues. The geometric insight gained through this global treatment of the Kepler problem by Fock in 1935 will be of essence in our subsequent discussions [2, 11].

Let us consider the momentum space Schrodinger equation of the Kepler problem in $\mathbb{R}^3$, which we obtain via a Fourier transform of the position space equation:

$$\Big(\frac{p^2}{2m} - E\Big)\phi(p) = \frac{k}{2\pi^2}\int_{\mathbb{R}^3} \frac{\phi(p')}{|p-p'|^2} d^3p'.$$

In general, Fock Quantization can be performed in arbitrary dimensions $\mathbb{R}^f$ where $f \geq 2$. We will proceed with the f-dimensional momentum space Schrdinger equation

$$\Big(\frac{p^2}{2m} - E\Big)\phi(p) = \frac{k}{\pi\Omega_f}\int_{\mathbb{R}^f}\frac{\phi(p')}{|p-p'|^{f-1}}d^f p',$$

where $\Omega_f = \frac{2\pi^{(f+1)/2}}{\Gamma[(f+1)/2]}$ is the area of the unit f-sphere S^f. Substituting in Ω_f and rearranging, we get

$$(p^2 + p_0^2)\phi(p) = \frac{mk\Gamma\big(\frac{f-1}{2}\big)}{\pi^{\frac{f+1}{2}}}\int_{\mathbb{R}^f}\frac{\phi(p')}{|p-p'|^{f-1}}d^f p', \tag{2.19}$$

where $p_0^2 = -2mH$. Let us take dimensionless Cartesian coordinates $x_i = p_i/p_0$ on the momentum space $\mathbb{R}^f$. The stereographic projection from $\mathbb{R}^f$ with coordinates x_i onto a unit f-sphere S^f embedded in $\mathbb{R}^{f+1}$ with coordinates (X_i, X_{f+1}), where $i = 1, 2, \cdots, f$, is given by

$$\begin{aligned} X_i &= \frac{2x_i}{x^2+1} = \frac{2p_i}{p^2+p_0^2}, \\ X_{f+1} &= \frac{x^2-1}{x^2+1} = \frac{p^2-p_0^2}{p^2+p_0^2}, \end{aligned} \tag{2.20}$$

where $\langle X, X\rangle = 1$. Since metric tensor of S^f is given by $g_{ij} = (\frac{2}{x^2+1})^2\delta_{ij}$ due to the first expression in Eq. (2.20), the surface element of S^f is

$$d\Omega_f = \left[\frac{2}{x^2+1}\right]^f d^f x = \left[\frac{2p_0}{p^2+p_0^2}\right]^f d^f p, \tag{2.21}$$

where in the last step we substituted in $x_i = p_i/p_0$. To obtain an expression for the denominator $|p-p'|^{f-1}$ inside the integral in terms of $|X - X'|$, we calculate

$$\begin{aligned} |X-X'|^2 &= \langle X, X\rangle + \langle X', X'\rangle - 2\langle X, X'\rangle \\ &= 2 - 2\frac{4\langle x, x'\rangle + (x^2-1)(x'^2-1)}{(x^2+1)(x'^2+1)} \\ &= \frac{4|x-x'|^2}{(x^2+1)(x'^2+1)} \\ &= \frac{4p_0^2}{(p^2+p_0^2)(p'^2+p_0^2)} \quad |p-p'|^2, \end{aligned}$$

where in the second step we used Eq. (2.20). Hence,

$$\frac{1}{|X-X'|^{f-1}} = \left[\frac{(p^2+p_0^2)(p'^2+p_0^2)}{4p_0^2}\right]^{\frac{f-1}{2}}\frac{1}{|p-p'|^{f-1}}. \tag{2.22}$$

Finally, we can remove all factors of p and p' in Eq. (2.19) if we define an appropriately rescaled wavefunction on S^f:

$$\Phi(X) = \left[\frac{p^2 + p_0^2}{2p_0^2}\right]^{\frac{f+1}{2}} \phi(p). \tag{2.23}$$

Plugging Eq. (2.21)-(2.23) into Eq. (2.19), we get

$$\frac{\Gamma\left(\frac{f-1}{2}\right)}{2\pi^{\frac{f+1}{2}}} \int_{S^f} \frac{\Phi(X')}{|X - X'|^{f-1}} d\Omega'_f = \frac{p_0}{mk}\Phi(X). \tag{2.24}$$

We have mapped $\mathbb{R}^f$ to S^f by changing to stereographic coordinates (X_i, X_{f+1}), so we are now integrating over the surface of S^f. The change of coordinates makes it apparent that the problem is symmetric under $SO(f+1)$. It will be useful to define the operator

$$\hat{\mathcal{L}}^{-1}\Phi(X) = \frac{\Gamma\left(\frac{f-1}{2}\right)}{2\pi^{\frac{f+1}{2}}} \int_{S^f} \frac{\Phi(X')}{|X - X'|^{f-1}} d\Omega'_f, \tag{2.25}$$

such that we see from

$$\hat{\mathcal{L}}^{-1}\Phi(X) = \frac{p_0}{mk}\Phi(X) \tag{2.26}$$

that $\Phi(X)$ is the eigenstate and mk/p_0 is the eigenvalue of $\hat{\mathcal{L}}$.

The convolution integral in Eq. (2.24) can also be obtained purely mathematically by considering homogeneous harmonic polynomials. A homogeneous harmonic polynomial $h_\lambda(X)$ of degree λ in $\mathbb{R}^{f+1}$ satisfies the following conditions: (1) harmonicity

$$\Delta_{\mathbb{R}^{f+1}} h_\lambda(X) = 0 \tag{2.27}$$

and (2) homogeneity

$$h_\lambda(X) = |X|^\lambda Y_\lambda\left(\tfrac{X}{|X|}\right), \tag{2.28}$$

where $Y_\lambda\left(\frac{X}{|X|}\right)$, called **spherical harmonics**, are homogeneous harmonic polynomials $h_\lambda(X)$ restricted to S^f. We recall that the spherical harmonic functions, generalized to $\mathbb{R}^{f+1}$, form a basis for the Hilbert space of square-integrable functions L^2 on S^f. Therefore, any function on S^f may be expressed as a linear combination of the spherical harmonics functions Y_λ. If we can obtain the convolution integral in Eq. (2.24) such that Y_λ is the eigenstate of $\hat{\mathcal{L}}$, we can compare the resulting eigenvalue to that of Eq. (2.26) to obtain the hydrogen energy levels.

To do this, let us first note that the kernel $\frac{1}{|X-X'|^{f-1}}$ in the integral of Eq. (2.24) is the Green's function of the Laplacian $\Delta_{\mathbb{R}^{f+1}}$ in $\mathbb{R}^{f+1}$:

$$\Delta_{\mathbb{R}^{f+1}} \frac{1}{|X - X'|^{f-1}} = -(f-1)\Omega_f \delta^{f+1}(X - X'). \tag{2.29}$$

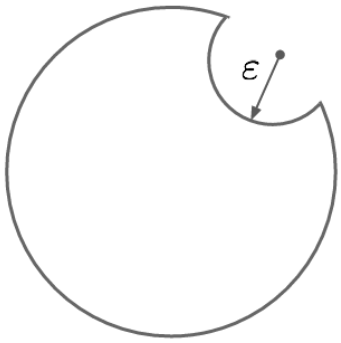

FIGURE 2.3: "Dented" sphere of integration

The kernel $\frac{1}{|X-X'|^{f-1}}$ is harmonic everywhere except at $X' = X$. Then, let us consider the surface of integration S_ϵ such that

$$S_\epsilon \equiv \{X' : |X'|^2 = 1, |X - X'| > \epsilon\} \cup \{X' : |X'|^2 \leq 1, |X - X'| = \epsilon\},$$

where X is a fixed point on S^f and X' is a variable. S_ϵ is essentially a "dented" S^f where X is placed outside the surface of integration S_ϵ via a shell centered at X with radius ϵ (Fig. 2.3). Green's second identity gives us

$$\int_{V(S_\epsilon)} \left[h_\lambda(X') \Delta \frac{1}{|X - X'|^{f-1}} - \frac{1}{|X - X'|^{f-1}} \Delta h_\lambda(X') \right] dV'_f$$
$$= \int_{S_\epsilon} \left[h_\lambda(X') \frac{d}{dn} \frac{1}{|X - X'|^{f-1}} - \frac{1}{|X - X'|^{f-1}} \frac{d}{dn} h_\lambda(X') \right] d\Omega'_f,$$

where $V(S_\epsilon)$ denotes the volume enclosed by S_ϵ, n is the outward normal to S_ϵ, and $\Delta \equiv \Delta_{\mathbb{R}^{f+1}}$. Since $h_\lambda(X')$ is harmonic and since S_ϵ does not contain X, Eq. (2.27) and Eq. (2.29) tell us that the integral over $V(S_\epsilon)$ disappears. Then, we have

$$0 = \int_{S_\epsilon} \left[h_\lambda(X') \frac{d}{dn} \frac{1}{|X - X'|^{f-1}} - \frac{1}{|X - X'|^{f-1}} \frac{d}{dn} h_\lambda(X') \right] d\Omega'_f. \tag{2.30}$$

The integral splits into two parts based on the region of integration. Let us treat each separately. The first part is over the shell centered at X with radius ϵ. If we take $\epsilon \to 0$, the shell becomes a hemispherical shell as S^f is locally flat. To solve the integral, we can complete the shell to a spherical shell around X and only take half of the result. Using Green's second identity over this region and halving, we get $\frac{f-1}{2}\Omega_f Y_\lambda(X)$ since $h_\lambda = Y_\lambda$ on S^f. The negative sign from Eq. (2.29) disappears if we recall that n is the inward normal in this region. The second part tends smoothly to an integral over S^f as

$\epsilon \to 0$. Due to homogeneity in Eq. (2.28),

$$\frac{d}{dn}h_\lambda(X')\Big|_{|X'|=1} = \lambda Y_\lambda(X'),$$

and on the S^f where $|X| = |X'| = 1$,

$$\frac{d}{dn}\frac{1}{|X-X'|^{f-1}}\Bigg|_{|X|=|X'|=1} = -\frac{f-1}{2}\frac{1}{|X-X'|^{f-1}}\Bigg|_{|X|=|X'|=1}. \tag{2.31}$$

Eq. (2.31) is intuitive if we recall the following special property of the inverse-square law for $f = 3$ (in $\mathbb{R}^3$): the density of field lines for an inverse-square force emanating from a point source also happens to decrease as inverse-square. Similarly in $\mathbb{R}^f$, the density decreases as $1/r^{f-1}$. For a rigorous derivation which includes the $\frac{f-1}{2}$ factor in the general case $f \geq 2$, refer to pg. 85 of Cordani [2]. Inserting the results into Eq. (2.30), we get

$$\frac{\Gamma\left(\frac{f-1}{2}\right)}{2\pi^{\frac{f+1}{2}}}\int_{S^f}\frac{Y_\lambda(X')}{|X-X'|^{f-1}}d\Omega'_f = \frac{2}{f-1+2\lambda}Y_\lambda(X). \tag{2.32}$$

Noticing that the left hand side of Eq. (2.32) can be expressed in terms of $\hat{\mathcal{L}}$,

$$\hat{\mathcal{L}}^{-1}Y_\lambda(X) = \frac{2}{f-1+2\lambda}Y_\lambda(X), \quad \lambda = 0, 1, 2, \cdots. \tag{2.33}$$

Comparing and rearranging the eigenvalues in Eq. (2.26) and Eq. (2.33), we have for the general case $f \geq 2$

$$H = -\frac{2mk^2}{[2\lambda + f - 1]^2}.$$

For the physical case $f = 3$, we have

$$H = -\frac{mk^2}{2(\lambda+1)^2} = -\frac{mk^2}{2n^2} = -\frac{m_e e^4}{2(4\pi\epsilon_0)^2\hbar^2}\frac{1}{n^2}, \tag{2.34}$$

where we identified $\lambda = 2j$ and we inserted the term $\hbar^2$ in the last expression. Again, we retrieved the correct energy levels for the hydrogen atom.

As in the previous section, let us look at the degeneracies that give rise to the n^2 term in the energy levels. An arbitrary homogeneous polynomial in $f+1$ variables (in $\mathbb{R}^{f+1}$) of degree λ depends on $\binom{\lambda+f}{\lambda}$ constants. Since the harmonicity condition in Eq. (2.27) gives $\binom{\lambda+f-2}{\lambda-2}$ constants, the number of independent spherical harmonics Y_λ with the same value of λ is

$$\binom{\lambda+f}{\lambda} - \binom{\lambda+f-2}{\lambda-2} = \frac{(\lambda+f-2)!(2\lambda+f-1)}{(f-1)!\lambda!}.$$

For $f = 3$, this reduces to $(\lambda + 1)^2 = n^2$. This assures us that we have found all the solutions of the eigenvalue problem on the sphere [10].

A similar procedure can be conducted with $H > 0$ (scattering) Kepler problems [12]. One maps the wavefunction to the f-hyperboloid H^f via an extended stereographic projection. The upper and lower hyperboloids, H^f_+ and H^f_-, describe the $k > 0$ (repulsive) and $k < 0$ (attractive) problem, respectively. The problem is then cast in a convolution integral equation similar to Eq. (2.24) that is symmetric under $SO(1, f)$. In obtaining the same integral using homogeneous harmonic polynomials, one uses hypergeometric functions, which form a basis for the Hilbert space of square-integrable functions L^2 on H^f, in place of spherical harmonics. Although of the same form as in the bound case, the energy eigenvalues for the scattering problem involve a complex λ, which is an analytic continuation of the λ in Eq. (2.34). Indeed, one expects this result, since the energy levels in the scattering problem are continuous.

Lastly, we would like to note the classical-quantum correspondence of the Kepler problem. In this chapter, we saw that the classical and quantum Kepler problems share the same natural manifold. On such a manifold, classical particle paths are geodesics and quantum wavefunctions are spherical harmonics. Therefore, the spherical harmonics must give rise to the geodesic flow on the manifold at length scales greater than the atomic. This can be done using a process called geometric quantization. However, this thesis will not cover geometric quantization, as the method is rather involved and will detract from the thesis. Interested readers can look to Ch. 9 of Cordani [2] for a thorough discussion.

Chapter 3

Classical Regularization

In physics, **regularization** is a method of dealing with divergent expressions. In quantum field theory, one does this in one of several ways, for example, by introducing a regulator ϵ to avoid infinities while carrying out calculations after which one takes the limit as $\epsilon \to 0$. In the Kepler problem, one encounters collision orbits, where the elliptical orbit degenerates into a straight line and the motion reaches the origin at a finite time t. For such orbits, the potential term in the Hamiltonian blows up, resulting in a singularity in the phase space and an incomplete space of possible orbits. In this chapter, we will consider the Moser method, where we embed the incomplete space of orbits into a complete one in order to remove the singularity due to collision orbits in the phase space. Alternatively, we also consider the Kustaanheimo-Stiefel map, where we show the equivalence of the 3d Kepler problem to the 4d harmonic oscillator via a canonical transformation. This turns a system with a singularity in phase space into an equivalent regularized system. The methods presented in this chapter go beyond regularization; they use the symmetry properties of the Kepler problem in order to recast the problem in a more natural light. It is assumed that readers are familiar with basics of differential geometry and Hamiltonian mechanics.[1]

3.1 Moser Method

In Section 2.1, we showed that the momentum space curves of the Kepler problem are geodesics on the manifolds S^3 or H^3. In the Moser method [1, 7], we will start from the (co-)geodesic flow on S^3 and H^3 and obtain the Hamiltonian K on the phase space, which is a reparametrization of the familiar Kepler Hamiltonian H. It turns out that switching to the parameter s of the motion given by K, which replaces the time parameter t of H, removes the singularity of the collision orbits. In the process, we obtain the expressions

[1] In particular, we use the symplectic manifold approach to Hamiltonian mechanics.

for the motion on the *phase space*, rather than simply the momentum space, of the Kepler problem by projecting the phase space of a free particle on S^3 and H^3.

If we are to consider the phase space of a particle on S^3 and H^3, it is helpful to cast the problem in the language of manifolds [4, 13]. In a d-dimensional Hamiltonian system, the motion at any point in time is given by its position, which is specified by the canonical coordinates (q^i, p_i) on the symplectic manifold. **Symplectic manifold** is a $2d$-dimensional phase space with a symplectic 2-form $\omega = dq^i \wedge dp_i$. The symplectic 2-form ω provides an identification between 1-forms and vector fields. For a function f and its corresponding vector field $\xi_f = \{\cdot, f\} = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}$, the following relation holds:

$$df = -\omega(\xi_f, \cdot). \tag{3.1}$$

If we are given the d-dimensional configuration manifold M, we can construct the phase manifold by considering the **cotangent bundle** T^*M to M. The cotangent bundle T^*M can be constructed by considering the cotangent space $T^*_m M$ at a point m on M, then taking the union of $T^*_m M$ for all $m \in M$. For example, the motion of a 1-dimensional rigid pendulum is restricted to the circle $M = S^1$ and, at each point m of the circle, the pendulum can have any real value of momentum $T^*_m M = \mathbb{R}^1$. Thus, the cotangent bundle of the system is given by the cylinder $T^*M = S^1 \times \mathbb{R}^1$. In fact, the cotangent bundle T^*M *is* the phase space of a particle whose motion occurs on M, and T^*M is a symplectic manifold with a symplectic 2-form ω.

Geodesics on the configuration manifold M are paths traced by free particles on M. Thus, the (co-)geodesic flow on M is described by the Hamilton's equations of motion given by the free Hamiltonian on T^*M

$$G(q^i, p_i) = \frac{1}{2}|p|^2, \tag{3.2}$$

where $|p|^2 = \sum_i p_i^2$ and $p_i \in T^*M$ is the conjugate momenta. Thus, geodesics are, in more precise terms, projections of curves induced by the geodesic flow onto the configurational manifold M. The **canonical 1-form** is defined on the cotangent space T^*M and is given by $\theta = p_i dq^i$. The canonical 1-form θ is preserved under canonical transformations, and the exterior derivative of θ gives the symplectic 2-form $\omega = d\theta$.

We will treat the geodesics flow on the manifolds for the bound state ($E < 0$) and scattering state ($E > 0$) problems separately. The concepts introduced in the Moser method will serve as an important prelude to Chapter 5, where we transform between the energy regimes by taking different cross-sections of a greater manifold.

3.1.1 Bound State

In the bound state ($E < 0$) problem, we consider the configuration manifold S^3 embedded in $\mathbb{R}^4$ [1, 7]. Let $X = (X_0, \vec{X}) \in \mathbb{R}^4$ be the configuration coordinates. Though X are physically projective momentum coordinates, we first take them to be configuration coordinates to obtain the cotangent bundle. We will perform a canonical exchange later. Then S^3 consists of all unit vectors $|X|^2 = 1$, where $|X|^2 = X \cdot X = X_0^2 + |\vec{X}|^2$. Taking $S^3_N = S^3 - \{N\}$ where $N = (1, \vec{0})$ is the north pole of S^3, we see that the stereographic projection $f : S^3_N \to \mathbb{R}^3$ from N gives

$$\vec{x} = \frac{\vec{X}}{1 - X_0} \tag{3.3}$$

for the coordinates $\vec{x} \in \mathbb{R}^3$. The inverse map $f^{-1} : \mathbb{R}^3 \to S^3_N$ is given by

$$X_0 = \frac{|\vec{x}|^2 - 1}{|\vec{x}|^2 + 1}, \tag{3.4}$$

$$\vec{X} = \frac{2\vec{x}}{|\vec{x}|^2 + 1}. \tag{3.5}$$

The stereographic projection thus provides an isomorphism between S^3_N and $\mathbb{R}^3$. We would like to find such a map between the cotangent bundles $T^*S^3_N$ and $T^*\mathbb{R}^3$. We embed the cotangent bundle $T^*S^3_N$ in $T^*\mathbb{R}^4$ and identify $T^*\mathbb{R}^4 = \mathbb{R}^4 \oplus \mathbb{R}^4$. Let us define $Y = (Y_0, \vec{Y}) \in \mathbb{R}^4$ as the conjugate momentum of $X \in \mathbb{R}^4$. Similarly, we identify that $T^*\mathbb{R}^3 = \mathbb{R}^3 \oplus \mathbb{R}^3$ and define $\vec{y} \in \mathbb{R}^3$ as the conjugate momentum of $\vec{x}$. We now impose the constraints

$$|X|^2 = X_0^2 + |\vec{X}|^2 = 1, \tag{3.6}$$

$$X \cdot Y = X_0 Y_0 + \vec{X} \cdot \vec{Y} = 0, \tag{3.7}$$

$$\frac{1}{2} d|X|^2 = X_0 dX_0 + \vec{X} \cdot d\vec{X} = 0. \tag{3.8}$$

In addition, we would like to have the canonical 1-forms on $T^*S^3_N$ and $T^*\mathbb{R}^3$ be equal to each other (i.e. coordinates are related by a canonical transformation):

$$Y \cdot dX = \vec{y} \cdot d\vec{x}. \tag{3.9}$$

Let us now determine the expression for $\vec{y}$ as a function of X and Y. We claim that

$$\vec{y} = (1 - X_0)\vec{Y} + Y_0 \vec{X}. \tag{3.10}$$

We can check Eq. (3.9) by showing that it satisfies Eq. (3.8). First, the exterior derivative of Eq. (3.3) is

$$d\vec{x} = \frac{1}{1 - X_0} d\vec{X} + \frac{dX_0}{(1 - X_0)^2} \vec{X}.$$

Taking the dot product with Eq. (3.9) gives

$$\begin{aligned}
\vec{y} \cdot d\vec{x} &= \vec{Y} \cdot d\vec{X} + (\vec{X} \cdot \vec{Y}) \frac{dX_0}{1 - X_0} + Y_0 \frac{\vec{X} \cdot d\vec{X}}{1 - X_0} + (\vec{X} \cdot \vec{X}) \frac{Y_0 dX_0}{(1 - X_0)^2} \\
&= \vec{Y} \cdot d\vec{X} - X_0 Y_0 \frac{dX_0}{1 - X_0} - X_0 Y_0 \frac{dX_0}{1 - X_0} + (1 + X_0) Y_0 \frac{dX_0}{1 - X_0} \\
&= \vec{Y} \cdot d\vec{X} + (1 - X_0) Y_0 \frac{dX_0}{1 - X_0} \\
&= Y_0 dX_0 + \vec{Y} \cdot d\vec{X} \\
&= Y \cdot dX, \quad \checkmark
\end{aligned}$$

where we used Eq. (3.6), (3.7), and (3.8) in the second step. Let us now determine the expression for Y as a function of $\vec{x}$ and $\vec{y}$. Consider the following expression gives us Y_0:

$$\begin{aligned}
\vec{x} \cdot \vec{y} &= (1 - X_0) \frac{\vec{X} \cdot \vec{Y}}{(1 - X_0)} + Y_0 \frac{\vec{X} \cdot \vec{X}}{1 - X_0} \\
&= -X_0 Y_0 + Y_0 (1 + X_0) \\
&= Y_0,
\end{aligned}$$

where we used Eq. (3.6) and (3.7) in the second step. Thus,

$$Y_0 = \vec{x} \cdot \vec{y}. \tag{3.11}$$

Now we can solve for $\vec{Y}$. Solving for $\vec{Y}$ from Eq. (3.10) and inserting Eq. (3.11), we get

$$\vec{Y} = \frac{1}{1 - X_0} \Big[\vec{y} - (\vec{x} \cdot \vec{y}) \vec{X} \Big].$$

We can use Eq. (3.3) and the result $\frac{1}{1 - X_0} = \frac{|\vec{x}|^2 + 1}{2}$ from Eq. (3.4) to get

$$\vec{Y} = \frac{1}{2} (|\vec{x}|^2 + 1) \vec{y} - (\vec{x} \cdot \vec{y}) \vec{x}. \tag{3.12}$$

Thus, we found the extended stereographic map and its inverse. Note that

$$|Y|^2 = Y_0^2 + |\vec{Y}|^2 = \left[\frac{1}{2} \big(|\vec{x}|^2 + 1 \big) \right]^2 |\vec{y}|^2.$$

The (co-)geodesic flow on S^3 is given by the free Hamiltonian F on T^*S^3, which we map to $T^*\mathbb{R}^3$ via stereographic projection:

$$F(\vec{x},\vec{y}) = \frac{1}{2}|X|^2|Y|^2 = \frac{1}{8}\big(|\vec{x}|^2+1\big)^2|\vec{y}|^2, \tag{3.13}$$

where upon inserting $|X|^2 = 1$ for the middle expression, we retrieve the familiar free Hamiltonian $F(X,Y) = \frac{1}{2}|Y|^2$ from Eq. (3.2). It is possible to verify that the Hamiltonian F satisfies the Hamilton's equations for the canonical coordinates (X,Y) and $(\vec{x},\vec{y})$.

Suppose we replace F by a new Hamiltonian $J = J(F)$. Then,

$$d(J(F)) = J'(F)\ dF.$$

Using Eq. (3.1) and canceling the symplectic 2-form ω, we write

$$\xi_{J(F)} = J'(F)\ \xi_F,$$

where $\xi_{J(F)}$ and ξ_F are Hamiltonian vector fields. The above relation implies that, for constant F and $J'(F) \neq 0$, the vector fields possess identical trajectories with velocities only differing by a constant term $J'(F)$. It is most convenient to choose $J(F)$ such that $J'(F) = 1$. Let us choose $J(F) = \sqrt{2F} - 1$, so that $J'(F) = \sqrt{\frac{1}{2F}}$. It follows that $J'(F) = 1$ for the submanifold of $T^*\mathbb{R}^3$ where $F = \frac{1}{2}$. Then, we get for the new Hamiltonian $J = J(F)$

$$J(\vec{x},\vec{y}) = \frac{1}{2}\big(|\vec{x}|^2+1\big)|\vec{y}| - 1. \tag{3.14}$$

The submanifold $F = \frac{1}{2}$ is then identical to the submanifold $J = 0$, and the two vector fields ξ_F and ξ_J coincide on the submanifolds. We can also verify that Hamiltonian J also satisfies Hamilton's equations for the canonical coordinates $(\vec{x},\vec{y})$. We now observe that

$$J = |\vec{y}|\ \Big(H + \frac{1}{2}\Big), \tag{3.15}$$

where $H(\vec{x},\vec{y}) = \frac{|\vec{x}|^2}{2} - \frac{1}{|\vec{y}|}$ and $|\vec{y}| \neq 0$. If we identify $\vec{p} = -\vec{x}$ and $\vec{q} = \vec{y}$, we recover

$$H(\vec{q},\vec{p}) = \frac{|\vec{p}|^2}{2} - \frac{1}{|\vec{q}|}, \tag{3.16}$$

the Kepler Hamiltonian. The identification above is intuitive if we recall that in Chapter 1 we mapped the momentum space to the 3-sphere. On the submanifold $H = -\frac{1}{2}$ identical to the submanifold $J = 0$, we can take the exterior derivative of Eq. (3.15) to get

$$dJ = |\vec{y}|\ d\Big(H + \frac{1}{2}\Big),$$

so

$$\xi_J = |\vec{y}|\, \xi_H.$$

From the definition of the Hamiltonian vector field ξ_H, we have $\xi_H = \{\cdot, H\} = \frac{d}{dt}$. The time parameter t is the independent variable corresponding to the Hamiltonian H. There also exists a parameter s corresponding to the Hamiltonian J. Then, t and s are related by $dt/ds = |\vec{y}|$. We know that H satisfies the Hamilton's equation for the canonical coordinates $(\vec{q}, \vec{p})$ with the independent variable t, and that J satisfies the same with coordinates $(\vec{x}, \vec{y})$ and variable s. Therefore, changing the independent variables from t to s amounts to changing the Hamiltonian from H to J. While the vector field ξ_H contains a singularity at $|\vec{y}| = 0$ in H, the vector field ξ_J is free of singularities. Since $H = -\frac{1}{2}$ is a constant, $|\vec{y}| = 0$ is equivalent to $|\vec{x}| = \infty$, and the singularity of ξ_H is at the north pole $N = (1, \vec{0})$ when interpreted on S^3. Thus, switching from t to s restores N on S^3 and compactifies the submanifold $H = -\frac{1}{2}$. The geodesics that pass through N correspond to collision orbits, where the motion reaches the origin at a finite time t. Here, we regularized the problem by switching to "fake" time s, where the motion only reaches the origin at infinite s.

By the way, we have limited our discussion so far to the equivalence of the geodesic flow on S^3 and the bound Kepler problem only for the submanifold $H = -\frac{1}{2}$ [2]. We can generalize this to any negative energy submanifold $H = -\frac{1}{2\lambda}$ if we rescale our variables

$$\vec{q} \mapsto \lambda^2 \vec{q}, \qquad \vec{p} \mapsto \frac{\vec{p}}{\lambda}, \qquad t \mapsto \lambda^3 t.$$

We can explicitly find the phase space orbits generated by the Moser method by considering a geodesic on the cross section of S^3 at $X_3 = 0$ [1]. The cross section is taken without loss of generality, since for a given geodesic, we can always rotate S^3 such that the geodesic lies on $X_3 = 0$. The most general such circle is given by

$$X_0 = \sin\alpha\cos s, \quad X_1 = \sin s, \quad X_2 = -\cos\alpha\cos s.$$

Differentiating with respect to the independent variable s, we get

$$Y_0 = -\sin\alpha\sin s, \quad Y_1 = \cos s, \quad Y_2 = \cos\alpha\sin s.$$

Identifying the eccentricity $e = \sin\alpha$ of the orbit and performing the extended stereographic map in Eq. (3.3) and Eq. (3.10),

$$\begin{aligned} x_1 &= \frac{\sin s}{1 - e\cos s}, \qquad & x_2 &= \frac{-\sqrt{1-e^2}\cos s}{1 - e\cos s} \\ y_1 &= \cos s - e, \qquad & y_2 &= \sqrt{1-e^2}\sin s. \end{aligned}$$

We thus have a phase space curve of a particle in a bound Kepler problem. Setting $\vec{p} = -\vec{x}$, we get

$$p_1^2 + \Big(p_2 - \frac{e}{\sqrt{1-e^2}}\Big)^2 = \Big(\frac{1}{\sqrt{1-e^2}}\Big)^2,$$

which agrees with the circular momentum space curve we saw in Eq. (2.7). In addition, the variable s is a well-known quantity in celestial mechanics called **eccentric anomaly**. Given the quantity $|\vec{y}| = 1 - e\cos s$ from the phase space curve, we can now explicitly determine the relationship between t and s:

$$t = \int_0^s |\vec{y}| ds' = \int_0^s (1 - e\cos s')ds',$$

or

$$t = s - e\sin s. \tag{3.17}$$

This transcendental equation is called the Kepler's equation. Through the Moser approach, we were thus not only able to regularize the Kepler problem by changing our independent variable from t to s, but also obtain the extended geometric picture of the phase space of the Kepler problem.

3.1.2 Scattering State

For the scattering state ($E > 0$) problem, we proceed in analogy to the bound state problem [14]. Let us consider the configuration manifold H^3 embedded in $\mathbb{R}^{1,3}$, and let $X = (X_0, \vec{X}) \in \mathbb{R}^{1,3}$ be the configuration coordinates. H^3 consists of all "unit" vectors $|X|^2 = 1$, where $|X|^2 = X_0^2 - |\vec{X}|^2$ under the new metric signature (1,3). The stereographic projection $f : H^3_N \to \mathbb{R}^3$, where $H^3_N = H^3 - N$ and $N = (1, \vec{0})$, is

$$\vec{x} = \frac{\vec{X}}{1 + X_0}$$

for the coordinates $\vec{x} \in \mathbb{R}^3$. The inverse map $f^{-1} : \mathbb{R}^3 \to H^3_N$ is

$$\begin{aligned} X_0 &= \frac{|\vec{x}|^2 + 1}{|\vec{x}|^2 - 1}, \\ \vec{X} &= \frac{2\vec{x}}{|\vec{x}|^2 - 1}. \end{aligned}$$

It is then straightforward to derive the following maps in analogy to the bound case:

$$\begin{aligned} \vec{y} &= (1 + X_0)\vec{Y} + Y_0\vec{X} \\ Y_0 &= -\vec{x}\cdot\vec{y} \\ \vec{Y} &= \frac{1}{2}(|\vec{x}|^2 - 1)\vec{y} - (\vec{x}\cdot\vec{y})\vec{x}. \end{aligned}$$

The norm $|Y|^2 = Y_0^2 - |\vec{Y}|^2$ is

$$|Y|^2 = -\frac{1}{4}(|\vec{x}|^2 - 1)^2|\vec{y}|^2.$$

and the free Hamiltonian F giving the geodesic flow on H^3 is

$$F(\vec{x}, \vec{y}) = \frac{1}{2}|X|^2|Y|^2 = -\frac{1}{8}(|\vec{x}|^2 - 1)^2|\vec{y}|^2.$$

This time, let us define the new Hamiltonian $J = \sqrt{-2F} - 1$ such that $J'(F) = \sqrt{\frac{1}{-2F}}$. Explicitly,

$$J(\vec{x}, \vec{y}) = \frac{1}{2}(|\vec{x}|^2 - 1)|\vec{y}| - 1. \tag{3.18}$$

Taking the convenient choice $J'(F) = 1$ so that the vector fields of the two Hamiltonians coincide, i.e. $\xi_J(F) = \xi_F$, we restrict ourselves to the constant energy submanifold $F = -\frac{1}{2}$ of $T^*\mathbb{R}^3$, which is equivalent to the submanifold $J = 0$ of the new Hamiltonian. It can be verified that the Hamiltonian J satisfies the Hamilton's equations for the canonical coordinates $(\vec{x}, \vec{y})$. We observe that

$$J = |\vec{y}|\Big(H - \frac{1}{2}\Big),$$

where $H(\vec{x}, \vec{y}) = \frac{|\vec{x}|^2}{2} - \frac{1}{|\vec{y}|}$ and $|\vec{y}| \neq 0$. Identifying $\vec{p} = -\vec{x}$ and $\vec{q} = \vec{y}$, we recover

$$H(\vec{q}, \vec{p}) = \frac{|\vec{p}|^2}{2} - \frac{1}{|\vec{q}|}, \tag{3.19}$$

the Kepler Hamiltonian. The submanifold $H = \frac{1}{2}$ of $T^*\mathbb{R}^3$ is identical to the submanifold $J = 0$. Indeed again, the regularizing parameter corresponding to Hamiltonian J is the eccentric anomaly s defined by the relation $dt/ds = |\vec{y}|$, where time t is the evolution parameter corresponding to Kepler Hamiltonian H. Again, we have inserted the pole N into H^3_N in our regularization to recover H^3, the complete space of orbits in the scattering Kepler problem.

We have restricted ourselves to the positive Kepler Hamiltonian $H = \frac{1}{2}$. We can generalize this to any positive energy submanifold $H = \frac{1}{2\lambda}$ if we rescale our variables

$$\vec{q} \mapsto \lambda^2\vec{q}, \qquad \vec{p} \mapsto \frac{\vec{p}}{\lambda}, \qquad t \mapsto \lambda^3 t.$$

in the same manner we have generalized the negative energy submanifold.

3.2 Kustaanheimo-Stiefel Method

The Kustaanheimo-Stiefel map regularizes the Kepler problem by showing the equivalence of the 3d bound/scattering Kepler problem and the 4d simple/inverted isotropic harmonic oscillator, respectively. The 4d isotropic harmonic oscillator is a regularized system. Though an unregularized system, we present the inverted harmonic oscillator since it is relevant to our discussion of the implications of Kepler-Lorentz duality. Though we will mainly claim the map and discuss some of its implications in this section, we will derive the Kustaanheimo-Stiefel map in Chapter 5. The equivalence of the two systems are also demonstrated in Chapter 5 by showing that the map is a canonical transformation.

The Kustaanheimo-Stiefel map $\mathbb{K} : T^*\mathbb{R}^4 \to T^*\mathbb{R}^3$ is given in the bound/scattering states of the Kepler problem. The $T^*\mathbb{R}^4$ in the domain of $\mathbb{K}$ denotes the cotangent bundle of the 4d harmonic oscillator, not that of the embedding space of 3-manifolds seen in the Moser method. We will present the inverse map and its implications via the Kepler-Lorentz duality in Chapter 5.

3.2.1 Bound State

In the bound state case [2, 15], let us take the coordinates x_i in momentum space $\mathbb{R}^3$. We can express x_i in terms of the coordinates z_μ in $\mathbb{R}^4$:

$$\begin{aligned} x_1 &= 2(z_1 z_3 + z_2 z_4), \\ x_2 &= 2(z_1 z_4 - z_2 z_3), \\ x_3 &= z_1^2 + z_2^2 - z_3^2 - z_4^2, \end{aligned} \tag{3.20}$$

from which it follows that

$$|x| = |z|^2. \tag{3.21}$$

In mathematical physics, this map alone is called the **Hopf map** $\mathcal{H} : S^3 \xrightarrow{S^1} S^2$, which amounts to representing S^3 on S^2 by attaching a S^1 fiber at each point of S^2. Hence, the Kustaanheimo-Stiefel map $\mathbb{K}$ is the canonical extension of the Hopf map $\mathcal{H}$ [14]. Properties of the new variables z_μ providing the intuition for the Hopf map will soon follow. Meanwhile, the coordinate space variables y_i in $T^*\mathbb{R}^3$ can be expressed in terms of z_μ, w_μ in $T^*\mathbb{R}^4$:

$$\begin{aligned} y_1 &= -\frac{z_1 w_3 + z_3 w_1 + z_2 w_4 + z_4 w_2}{|z|^2}, \\ y_2 &= \frac{z_2 w_3 + z_3 w_2 - z_1 w_4 - z_4 w_1}{|z|^2}, \\ y_3 &= \frac{z_3 w_3 + z_4 w_4 - z_1 w_1 - z_2 w_2}{|z|^2}. \end{aligned} \tag{3.22}$$

The coordinates of the domain possess the constraint

$$z_1 w_2 - z_2 w_1 + z_3 w_4 - z_3 w_4 = 0. \tag{3.23}$$

Since $\dim T^*\mathbb{R}^4 = 8$ and due to the constraint in Eq. (3.23), the map $\mathbb{K}$ must have a 1-dimensional kernel, which is the space in the domain that maps to 0 in the image. The map $\mathbb{K}$ is not one-to-one, since all pairs z', w' related to z, w by

$$\begin{pmatrix} z'_1 \\ z'_2 \\ z'_3 \\ z'_4 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \tag{3.24}$$

and

$$\begin{pmatrix} w'_1 \\ w'_2 \\ w'_3 \\ w'_4 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \tag{3.25}$$

are mapped to the same pair x_i, y_i for every φ. This gives the kernel. The S^1 "gauge" attached to every point of S^2 is exactly this kernel for the Hopf map taking $z_\mu \mapsto x_i$.

Now, let us take z_μ as coordinates and w_μ as conjugate momenta on the phase space $T^*\mathbb{R}^4$. We assume that z_μ, w_μ and x_i, y_i are related by a canonical transformation, which we will show in Chapter 5. Performing a canonical exchange, plugging the results into the new Hamiltonian J in Eq. (3.14) from the Moser method, and after considerable algebra, we get

$$J(z_\mu, w_\mu) = \frac{1}{2}(|z|^2 + |w|^2) - 1. \tag{3.26}$$

This is the Hamiltonian of the 4d isotropic harmonic oscillator. We have shown the equivalence of 3d Kepler problem and the 4d harmonic oscillator under a canonical transformation $\mathbb{K}$.

3.2.2 Scattering State

It is straightforward from the bound state case to consider the scattering state problem. The map taking $x_i \mapsto z_\mu$ is exactly the same as in the bound state case in Eq. (3.20), so we still have the Hopf map $\mathcal{H}$. The expression of y_i in terms of the pair z_μ, w_μ differs by a minus sign in each component:

$$\begin{aligned} y_1 &= \frac{z_1w_3 + z_3w_1 + z_2w_4 + z_4w_2}{|z|^2}, \\ y_2 &= \frac{z_1w_4 + z_4w_1 - z_2w_3 - z_3w_2}{|z|^2}, \\ y_3 &= \frac{z_1w_1 + z_2w_2 - z_3w_3 - z_4w_4}{|z|^2}. \end{aligned} \tag{3.27}$$

The constraint and the kernel relation in Eq. (3.23), (3.24), and (3.25) remain same as well. After conducting the same procedure as above, we insert our expressions into the Hamiltonian J of the scattering state Moser map in Eq. (3.18), which gives

$$J(z_\mu, w_\mu) = \frac{1}{2}(|z|^2 - |w|^2) - 1, \tag{3.28}$$

due to the negative sign in J. This is the Hamiltonian for the 4d inverted harmonic oscillator. Given that map $\mathbb{K}$ is a canonical transformation, this shows the equivalence of the 3d scattering state Kepler problem and the 4d inverted harmonic oscillator.

Chapter 4

The Kepler-Lorentz Duality

In Chapter 2, we saw that the scattering state Kepler problem, also known as Rutherford (Coulomb) scattering, is symmetric under the action of the Lorentz group $SO(1,3)$.[1] In the algebraic approach, we found that the Poisson bracket relations of the six generators L_i and $\bar{A}_i$ form a Lie algebra isomorphic to that of $SO(1,3)$. In the geometric approach, we saw that the momentum space curves of the scattering Kepler problem, which we found to be arcs of circles, mapped to geodesics on H^3, an invariant manifold of $SO(1,3)$. In this chapter, we merge the two approaches by finding the explicit form of the transformations, induced by the generators of $so(1,3)$, that map geodesics to geodesics on H^3.

While it is clear that physical transformations corresponding to the generators L_i are rotations, the transformations corresponding to $\bar{A}_i$ are less intuitive. Here, we take a hint from special relativity, which shares the same symmetry under the Lorentz group $SO(1,3)$ as the scattering-state Kepler problem. We can thus predict that the transformations of $\bar{A}_i$ will resemble those of boosts in relativity—the transformations between inertial reference frames related by relative velocities. We proceed with this analogy in mind. In constructing the analogous transformations in the scattering Kepler problem, we find that the hyperbolic "boosts" generated by $\bar{A}_i$ are expressible in terms of familiar quantities in scattering problems. We introduce new scattering variables in the velocity space to express the explicit boost representations of $SO(1,3)$. Remarkably, this allows us to cast the scattering-state Kepler problem in a form that is in an exact correspondence wit the relativistic free particle kinematics of special relativity. We call this correspondence the **Kepler-Lorentz duality**. We also find that physical interpretations of boost transformations that belong to the group $SL(2,\mathbb{C})$, a double cover of $SO(1,3)$, come out naturally from the construction. In addition to the transformations,

[1] The scattering state Kepler problem also includes attractive potential problems for which $E > 0$, but we limit our discussion to the repulsive case. The actual Rutherford scattering experiment involved scattering between alpha particles and the nuclei of the gold atom, which is indeed a repulsive Kepler problem.

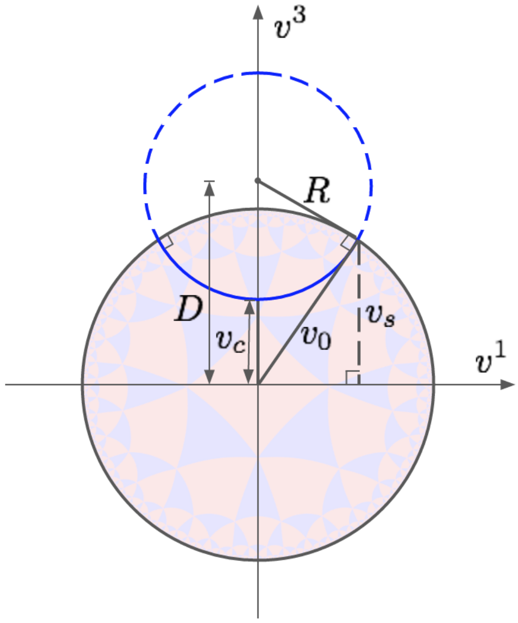

FIGURE 4.1: Poincaré disk (gray) in velocity space with a geodesic (solid blue)

we present the four-vectors of the scattering-state Kepler problem that are in correspondence to the velocity, momentum, and position 4-vectors of special relativity. As we construct the duality, it will become clear why we are first discussing the problem in velocity space rather than momentum space as we have done in previous chapters.

4.1 Geometric Context

Prior to further discussion, let us develop a geometric understanding of the scattering-state Kepler problem. This will be necessary to understand the motivation behind the new scattering variables.

In the scattering-state Kepler problem, the kinetic and potential terms of the Hamiltonian are both positive definite, and the potential term disappears as $r \to 0$. Thus, the asymptotic states of the scattered particle in velocity space must be those of a free particle and lie on the boundary of a 3-dimensional ball of radius $v_0 = \sqrt{\frac{2H}{m}}$. In the repulsive case, to which we limit our discussion, the speed of the particle decreases as it approaches the force-center and then increases to the limiting value as it recedes from it. The velocity of the scattered particle is hence always less than or equal to that in its asymptotic state, and its entire motion in velocity space lies inside the ball of radius v_0 centered at the origin. In Chapter 2, we saw that the velocity space curve of a scattering Kepler problem is an arc of a circle with a displaced center, where the arc lies inside the disk of radius v_0. We treated the problem on a 2-dimensional disk, since the angular momentum conservation allows for the planar treatment of the problem. For the Kepler problem in 3-dimensions, we consider geodesics on a 2-sphere in $\mathbb{R}^3$ with a

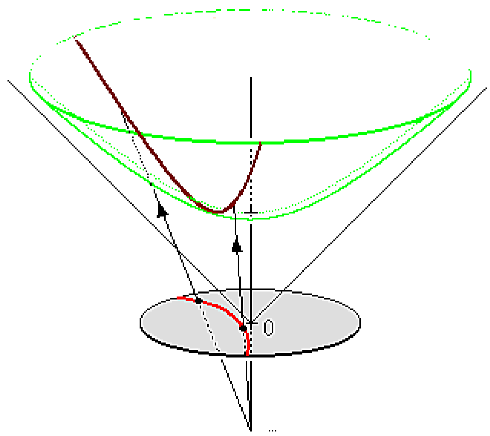

FIGURE 4.2: Stereographic projection between geodesics on $\mathcal{P}$ and on H^3

displaced center. The arc of such a geodesic that lies inside a 3-ball of radius v_0 at the origin is the velocity space trajectory of the scattering-state Kepler problem. This is a trivial generalization of picture seen in Chapter 2. Now, we claim that the geodesic and the ball intersect at orthogonal angles. We can prove this easily by verifying that the Pythagorean theorem holds between the radius of the ball v_0, the radius of the sphere R, and the displacement of the sphere center from the origin D (Fig. 4.1). Recall from Chapter 2 that $R = \frac{k}{L}$ and $D = \frac{A}{mL}$. Indeed,

$$R^2 + v_0^2 = \frac{m^2k^2 + 2mHL^2}{m^2L^2} = \left(\frac{A}{mL}\right)^2 = D^2.$$

It turns out that the ball in which the velocity space motion is contained can be identified with a model of hyperbolic geometry called the **Poincaré ball** $\mathcal{P}$. On the Poincaré ball $\mathcal{P}$ of radius v_0, the metric tensor is given by

$$ds^2 = \frac{4\,(dv)^2}{\left[1 - (\frac{v}{v_0})^2\right]^2}, \tag{4.1}$$

where v^i $(i = 1, 2, 3)$ are the Cartesian coordinates of $\mathbb{R}^3$ in which $\mathcal{P}$ is embedded. We have also written $v = \sqrt{v^i v_i}$. Geodesics on $\mathcal{P}$ are arcs of circles orthogonal to the boundary of $\mathcal{P}$. We have thus identified the velocity space curves of the scattering Kepler problem as geodesics on $\mathcal{P}$ [16]. As seen in Chapter 3, there is a stereographic map from $\mathcal{P}$ in $\mathbb{R}^3$ to the upper sheet of the 3-hyperboloid H^3 in $\mathbb{R}^{1,3}$, where the geodesics on $\mathcal{P}$ map to geodesics on H^3. This map, from $\mathcal{P}$ of radius v_0 to H^3 of the form

$$(V^0)^2 - (V^1)^2 - (V^2)^2 - (V^3)^2 = v_0^2, \tag{4.2}$$

is given by

$$\frac{V^0}{v_0} = \frac{1 + (\frac{v}{v_0})^2}{1 - (\frac{v}{v_0})^2}, \qquad \frac{V^i}{v_0} = \frac{2(\frac{v^i}{v_0})}{1 - (\frac{v}{v_0})^2}, \tag{4.3}$$

where the 4-velocity V^μ ($\mu = 0, 1, 2, 3$) are the Cartesian coordinates of $\mathbb{R}^{1,3}$ in which H^3 is embedded. Geodesics on H^3 are the intersections of H^3 with a plane containing the origin, and the "boost" transformations generated by $\bar{A}_i$ map geodesics to geodesics on H^3 by changing the slope of such a plane (Fig. 4.2). Thus, on $\mathcal{P}$, the generators $\bar{A}_i$ transform the R and D of the 2-sphere on which the geodesic curve lies.

From a group theoretic point of view, it is only natural at this point to fix the direction of the LRL vector $\mathbf{A}$ as $\bar{A}_i$ generate boosts in a particular direction. This is physically equivalent to fixing each of the scattering trajectories such that all trajectories are symmetric about a particular axis.

4.2 Scattering in New Variables

Let us consider the quantity v_c, the velocity of the scattered particle at the closest point of approach to the force center. That is, v_c is the minimum radial distance from the origin to a geodesic on the Poincaré ball $\mathcal{P}$. The quantity v_c is one of the few natural quantities to consider given the symmetries of the velocity-space particle trajectory. It will turn out that v_c is important quantity in the duality and in the boost transformations of spinors. At the closest point of approach, the LRL vector $\mathbf{A}$ is in the same direction as the position vector $\mathbf{r}$, and familiar scattering quantities such as eccentricity ε, impact parameter b, and the scattering angle θ are more intuitively obtained in terms of v_c. After we obtain the scattering quantities, we will find the relationship between v_s and v_c to express them in terms of v_s.

The eccentricity ε is related to $\mathbf{A}$ by

$$\varepsilon^2 = \left(\frac{A}{mk}\right)^2 = 1 + \frac{2EL^2}{mk^2}.$$

At the closest point of approach, the position and velocity vectors are orthogonal. The angular momentum is thus $L = mv_c r_c$, where r_c is the closest position of the scattering trajectory to the force center. Also, solving for r_c in the total energy $H = \frac{1}{2}mv_c^2 + \frac{k}{r_c}$

gives $r_c = \frac{2k}{m}\frac{1}{v_0^2 - v_c^2}$. The angular momentum is then $L = \frac{2kv_c}{v_0^2 - v_c^2}$. Thus,

$$\begin{aligned}
\varepsilon^2 &= 1 + \frac{v_0^2}{k^2}\Big(\frac{2kv_c}{v_0^2 - v_c^2}\Big)^2 \\
&= 1 + \frac{4v_0^2 v_c^2}{v_0^4 + v_c^4 - 2v_0^2 v_c^2} \\
&= \left(\frac{v_0^2 + v_c^2}{v_0^2 - v_c^2}\right)^2.
\end{aligned}$$

where we substituted $v_0 = \sqrt{\frac{2E}{m}}$ and L in the first line. It follows that

$$\varepsilon = \frac{1 + (\frac{v_c}{v_0})^2}{1 - (\frac{v_c}{v_0})^2}. \tag{4.4}$$

We can also obtain the impact parameter b in terms of v_c by comparing the components of $\mathbf{A}$ at the closest point of approach to the force center and at the asymptotic state. As $\mathbf{A}$ is an integral of motion, the sum of the components must remain invariant. Since the potential is repulsive, we take $k \to -k$, and $\mathbf{A}$ becomes

$$\mathbf{A} = m(\mathbf{v} \times \mathbf{L}) + mk\hat{\mathbf{r}}.$$

At the closest point of approach, $\mathbf{r}$ and $\mathbf{v}$ are orthogonal and it follows that $m(\mathbf{v} \times \mathbf{L}) = m^2(\mathbf{v} \times (\mathbf{r} \times \mathbf{v})) = (m^2 v_c^2 r_c)\hat{\mathbf{r}}$. The LRL vector $\mathbf{A}$ is thus given by

$$\mathbf{A} = (m^2 v_c^2 r_c + mk)\hat{\mathbf{r}}.$$

At the asymptotic state of scattering (i.e. at $t = \infty$), $L = mv_0 b$ and $\mathbf{A}$ is given by

$$\mathbf{A} = (m^2 v_0^2 b)\hat{\mathbf{n}}' + (mk)\hat{\mathbf{r}}',$$

where the unit vectors $\hat{\mathbf{n}}'$ and $\hat{\mathbf{r}}'$ point in orthogonal directions. Comparing $\mathbf{A}$ at the two points and using the Pythagorean theorem gives

$$\begin{aligned}
(m^2 v_0^2 b)^2 + (mk)^2 &= (m^2 v_c^2 r_c + mk)^2 \\
mv_0^4 b^2 &= mv_c^4 r_c^2 + 2kv_c^2 r_c.
\end{aligned}$$

Solving for b^2 and simplifying,

$$\begin{aligned}
b^2 &= r_c^2\Big(\frac{v_c}{v_0}\Big)^4 + \frac{2k}{mv_0^2} r_c\Big(\frac{v_c}{v_0}\Big)^2 \\
&= \frac{4k^2}{m^2 v_0^4}\frac{v_c^4}{(v_0^2 - v_c^2)^2} + \frac{4k^2}{m^2 v_0^4}\frac{v_c^2}{(v_0^2 - v_c^2)} \\
&= \Big(\frac{2k}{mv_0^2}\Big)^2 \frac{v_c^2}{(v_0^2 - v_c^2)}\left[\frac{v_c^2}{(v_0^2 - v_c^2)} + 1\right] \\
&= \Big(\frac{2k}{mv_0^2}\Big)^2 \frac{v_0^2 v_c^2}{(v_0^2 - v_c^2)^2} \\
&= \Big(\frac{2k}{mv_0^2}\Big)^2 \frac{(\frac{v_c}{v_0})^2}{\big[1 - (\frac{v_c}{v_0})^2\big]^2},
\end{aligned}$$

where we substituted for r_c in the second line. It follows that

$$b = \frac{k}{mv_0^2}\frac{2(\frac{v_c}{v_0})}{1 - (\frac{v_c}{v_0})^2}. \tag{4.5}$$

A mathematical property of hyperbolae directly relates the eccentricity ε to the scattering angle θ by $\varepsilon = \csc(\theta/2)$. The substitution of ε and a direct calculation gives

$$\theta = 2 \arcsin\left[\frac{1 - (\frac{v_c}{v_0})^2}{1 + (\frac{v_c}{v_0})^2}\right]. \tag{4.6}$$

One may have noticed that the form of the eccentricity ε in Eq. (4.4) and the impact parameter b in Eq. (4.5) closely resembles those of the stereographic map to the coordinates V^μ in Eq. (4.3). In our discussion of the duality, we will find that this is no coincidence.

Let us now introduce another variable v_s. Physically, the quantity v_s is the asymptotically invariant component of the scattered particle's velocity. On the Poincaré ball $\mathcal{P}$, it is the Cartesian component that remains unchanged at both ends of a geodesic. Note that the quantity v_s is defined with the Euclidean, rather than hyperbolic, metric on the velocity space (i.e. v_s is not infinite). The variable v_s is another natural quantity to consider given the symmetries of the velocity-space particle trajectory. Let us find the relationship between v_c and v_s. Fig. 4.1 shows the various quantities on $\mathcal{P}$. Given this picture, it is rather straightforward to derive the relationship between v_s and v_c. We notice the trigonometric relations $\cos\varphi = v_s/v_0$ and $\tan\varphi = R/v_0$, where φ is the angle between the v^3-axis and the solid gray line on $\mathcal{P}$ denoting the Euclidean radius v_0. We solve for R, which gives $R = v_0 \tan\varphi = (\frac{v_s}{v_0})^{-1}\sqrt{v_0^2 - v_s^2}$. We also see that $D = v_c + R$.

From the Pythagorean identity $D^2 = v_0^2 + R^2$ verified above, we get

$$\begin{gathered}(v_c + R)^2 = v_0^2 + R^2 \\ \Big(\frac{v_c}{v_0}\Big)^2 + \frac{2R}{v_0}\Big(\frac{v_c}{v_0}\Big) - 1 = 0\end{gathered}$$

Solving for v_c/v_0 and substituting R, we find after some algebra

$$\frac{v_c}{v_0} = \frac{1 - \sqrt{1 - (\frac{v_s}{v_0})^2}}{(\frac{v_s}{v_0})} = \frac{(\frac{v_s}{v_0})}{1 + \sqrt{1 - (\frac{v_s}{v_0})^2}}. \tag{4.7}$$

Also, inverting Eq. (4.7) gives

$$\frac{v_s}{v_0} = \frac{2(\frac{v_c}{v_0})}{1 + (\frac{v_c}{v_0})^2}. \tag{4.8}$$

Let us define the quantity $\delta = \sqrt{1 - (\frac{v_s}{v_0})^2}$ such that $\big(\frac{v_c}{v_0}\big)^2 = \frac{1-\delta}{1+\delta}$ for simplicity in computation.

We are now ready to express the scattering quantities in terms of v_s/v_0. Replacing v_c/v_0 in Eq. (4.4) gives for the eccentricity ε

$$\begin{aligned}\varepsilon &= \frac{1 + (\frac{v_c}{v_0})^2}{1 - (\frac{v_c}{v_0})^2} \\ &= \Big(1 + \frac{1-\delta}{1+\delta}\Big)\Big(1 - \frac{1-\delta}{1+\delta}\Big)^{-1} \\ &= \delta^{-1},\end{aligned}$$

and hence

$$\varepsilon = \frac{1}{\sqrt{1 - (\frac{v_s}{v_0})^2}}. \tag{4.9}$$

For the impact parameter b in Eq. (4.5),

$$\begin{aligned}b^2 &= \Big(\frac{2k}{mv_0^2}\Big)^2 \frac{(\frac{v_c}{v_0})^2}{\big[1 - (\frac{v_c}{v_0})^2\big]^2} \\ &= \Big(\frac{2k}{mv_0^2}\Big)^2 \Big(\frac{1-\delta}{1+\delta}\Big)\Big(1 - \frac{1-\delta}{1+\delta}\Big)^{-2} \\ &= \Big(\frac{2k}{mv_0^2}\Big)^2 \Big(\frac{1-\delta}{1+\delta}\Big)\Big(\frac{1+\delta}{2\delta}\Big)^2 \\ &= \Big(\frac{2k}{mv_0^2}\Big)^2 \Big(\frac{1-\delta^2}{4\delta^2}\Big) \\ &= \Big(\frac{k}{mv_0^2}\Big)^2 \ \frac{(\frac{v_s}{v_0})^2}{1 - (\frac{v_s}{v_0})^2}.\end{aligned}$$

Taking the square root, we get

$$b = \frac{k}{mv_0^2} \frac{(\frac{v_s}{v_0})}{\sqrt{1-(\frac{v_s}{v_0})^2}}. \tag{4.10}$$

Finally, the scattering angle θ in Eq. (4.6) is easily found to be

$$\theta = 2\cos^{-1}\left(\frac{v_s}{v_0}\right). \tag{4.11}$$

One can verify the expressions for ϵ, b, and θ by calculating the Rutherford scattering cross section in terms of v_s/v_0, then using their expressions to retrieve the known cross section.

4.3 Kepler-Lorentz Duality

The expressions for ε and b in Eq. (4.9) and (4.10) in the scattering-state Kepler problem are highly reminiscent of Lorentz transformations in special relativity. From the explicit forms of the scattering quantities, we derive the expressions of the corresponding boost transformations, analogous to that of Lorentz transformations, using the metric tensor of the Poincaré ball $\mathcal{P}$. Then, we derive the expressions of various four-vectors in the Kepler problem and compare the results to analogous ones in relativity.

4.3.1 Boost Transformations via SO(1,3)

We begin by considering a v^1–v^3 plane of the Poincaré ball $\mathcal{P}$ in Fig. 4.1, without loss of generality. The upper indices denote the index, not a power. At least in this chapter, we denote the power of an indexed quantity with a parenthesis, i.e. $(v)^3$. The metric tensor on $\mathcal{P}$, in Eq. (4.1), is

$$ds^2 = \frac{4\ (dv)^2}{\left[1-(\frac{v}{v_0})^2\right]^2},$$

In the end, we would like to express the resulting quantity in terms of v_s. However, it is easier to first consider integrating over the metric just along the v^3-axis, without loss of generality, from the origin to v_c. The expression in terms of v_s will follow from this consideration. The metric tensor thus becomes

$$ds = \frac{2\ d(v^3)}{1-(\frac{v^3}{v_0})^2}.$$

We can now integrate to get the distance to v_c on $\mathcal{P}$:

$$s \;=\; \int_0^{v_c} \frac{2\ d(v^3)}{1-(\frac{v^3}{v_0})^2} \;=\; 2v_0 \tanh^{-1}\left(\frac{v_c}{v_0}\right).$$

Inverting, we obtain the following:

$$\frac{v_c}{v_0} = \tanh\left(\frac{s}{2v_0}\right).$$

Now, notice that the relation between v_s and v_c in Eq. (4.8) is the angle addition formula for hyperbolic tangents, in the special case when the angles being added are equal. The quantity v_s is thus twice as big as v_c in the relativistic sense. Plugging the above relation into Eq. (4.8), we get

$$\frac{v_s}{v_0} = \frac{2\tanh\left(\frac{s}{2v_0}\right)}{1+\tanh^2\left(\frac{s}{2v_0}\right)} = \tanh\left(\frac{s}{v_0}\right).$$

Let us define **rapidity** $\phi \equiv \frac{s}{v_0}$, following the analogy from special relativity. In relativity, rapidity ϕ is the additive parameter of boost transformations among inertial frames. Then,

$$\tanh\phi = \frac{v_s}{v_0}. \tag{4.12}$$

Also defining $\zeta \equiv \frac{v_s}{v_0}$, it follows in a straightforward manner from trigonometric relations that

$$\cosh\phi = \frac{1}{\sqrt{1-\left(\frac{v_s}{v_0}\right)^2}} = \varepsilon, \tag{4.13}$$

$$\sinh\phi = \frac{\left(\frac{v_s}{v_0}\right)}{\sqrt{1-\left(\frac{v_s}{v_0}\right)^2}} = \zeta\varepsilon = \frac{mv_0^2 b}{k}. \tag{4.14}$$

Indeed, $\cosh\phi$ and $\sinh\phi$ are the matrix elements of the representations of the Lorentz group $SO(1,3)$ corresponding to boost transformations. We can thus specify the quantities of correspondence in the boost transformations of the scattering-state Kepler problem and special relativity, which is evident from Eq. (4.13) and (4.14):

$$\begin{array}{ccc} \textbf{Scattering Kepler} & & \textbf{Special Relativity} \\ \zeta = \dfrac{v_s}{v_0} & \longleftrightarrow & \beta = \dfrac{v}{c} \\ \varepsilon = \dfrac{1}{\sqrt{1-\left(\frac{v_s}{v_0}\right)^2}} & \longleftrightarrow & \gamma = \dfrac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \end{array} \tag{4.15}$$

It follows that $v_s \leftrightarrow v$ and $v_0 \leftrightarrow c$. Given the duality, we can show the form of boost transformations in the two contexts explicitly. In special relativity, the boost transformations of the 4-velocity U^μ along the U^3-axis is given by

$$\begin{pmatrix} U'^0 \\ U'^1 \\ U'^2 \\ U'^3 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} U^0 \\ U^1 \\ U^2 \\ U^3 \end{pmatrix}.$$

In the scattering-state Kepler problem, the same transformations of the corresponding 4-velocity V^μ, defined in Eq. (4.2) and (4.3), along the V^3-axis is given by

$$\begin{pmatrix} V'^0 \\ V'^1 \\ V'^2 \\ V'^3 \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 & 0 & -\varepsilon\zeta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\varepsilon\zeta & 0 & 0 & \varepsilon \end{pmatrix} \begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix}.$$

Therefore, U^μ and V^μ have same transformation properties and the new variable v_s/v_0 yields the duality between boost transformations.

4.3.2 Four-vectors

Four-vectors X^μ are central objects in special relativity and Lorentz-covariant theories that transform under the action of the Lorentz group $SO(1,3)$ as

$$X^\mu \to X'^\mu = \Lambda^\mu{}_\nu X^\nu$$

where $\Lambda \in SO(1,3)$. Here, we show the equivalence of various four-vectors in the scattering Kepler problem and special relativity under the boost duality relations in Eq. (4.15).

Let us first make geometrically precise the objects of correspondence which are subject to boost transformations in the two contexts. In special relativity, the velocity of inertial frames is the object on which boost transformations act. Each point on the velocity-space Poincaré ball $\mathcal{P}$ represents a particular inertial frame, and boost transformations take one point on $\mathcal{P}$ to another. The Poincaré ball $\mathcal{P}$ is the stereographic projection of the invariant H^3 in the Minkowski velocity space $\mathbb{R}^{1,3}$. In the scattering Kepler problem, we have so far limited our discussion on $\mathcal{P}$ to geodesics rather than points. On $\mathcal{P}$, there is an isomorphism between geodesics and points, modulo rotations about an axis given by v_c. That is, the location of the point v_c on $\mathcal{P}$ uniquely specifies the entire geodesic curve, up to a rotation of $\mathcal{P}$ about the axis on which v_c lies.[2] With some thought, this fact is apparent from Fig. 4.1. Therefore, in the scattering problem, the velocity of the scattered particle at the closest point of approach, $|v| = v_c$, is the

[2]Geodesics with v_c at $\{0\}$ are head-on collision curves and are specified up to rotation of the ball $\mathcal{P}$.

object on which boost transformations act. Therefore, when checking the duality of four-vectors, we must evaluate the four-vector components of the scattering problem at $|v| = v_c$, although its general expression may be used to specify the whole curve. We have thus constructed the duality between points on $\mathcal{P}$ in special relativity and the scattering Kepler problem.

But wasn't the frame velocity v in special relativity dual to the asymptotically invariant component of the velocity v_s in the scattering-state Kepler problem, rather than being dual to the velocity v_c at the closest point of approach to the target? This statement is true, but there is no conflict. For any value of the eccentricity ε, there is an isomorphism between v_s and v_c given by Eq. (4.7) and (4.8). Let us take $\mathbf{v_s}$ and $\mathbf{v_c}$ as vectors pointing from the origin to the quantities v_s and v_c along the symmetry axis of the velocity-space curve. Due to the symmetry of the curve, one cannot build a consistent algorithm to construct an isomorphism (up to rotation about the symmetry axis of the curve) between the space of vectors and of curves unless the vector points along the symmetry axis of the curve. Since only the endpoint of the vector $\mathbf{v_c}$ lies on the curve, we evaluate the particle velocity at $|v| = v_c$ for consistency, and there is no conflict. In the more elegant spinor representations considered below, we show that the dualities of boost representations and Hermitian forms of 4-vectors are naturally expressed with the use of v_c only, rather than both v_s and v_c.

Let us check the dualities of four-vectors. For the 4-velocities, we can check that the norm-squared of the components of 4-velocities are equal under the boost dualities in Eq. (4.15). Taking the norm-square of component U^0 of the relativistic 4-velocity U^μ, we get

$$U^0 U_0 \;=\; \frac{dx^0}{d\tau}\frac{dx_0}{d\tau} \;=\; c^2\Big(\frac{dt}{d\tau}\Big)^2 \;=\; c^2\gamma^2 \;\leftrightarrow\; v_0^2\varepsilon^2, \tag{4.16}$$

where τ is the proper time. In the last step, we used the duality relation in Eq. (4.15). If we contract V^0 of the scattering Kepler 4-velocity V^μ with itself and evaluate at $|v| = v_c$, we get

$$V^0 V_0\big|_{|v|=v_c} \;=\; v_0^2\left[\frac{1+(\frac{v_c}{v_0})^2}{1-(\frac{v_c}{v_0})^2}\right]^2 \;=\; v_0^2\varepsilon^2, \tag{4.17}$$

where in the last step we used Eq. (4.4). We thus verify from Eq. (4.16) and (4.17) that the 0-th components U^0 and V^0 are equal up to orthogonal transformations. The contraction of U^i with itself gives

$$U^i U_i \;=\; c^2\Big[\Big(\frac{dt}{d\tau}\Big)^2 - 1\Big] \;=\; c^2(\gamma^2-1) \;\leftrightarrow\; v_0^2(\varepsilon^2-1), \tag{4.18}$$

where the first step follows from the 4-vector contraction

$$\eta_{\mu\nu}U^\mu U^\nu \;=\; U^0U_0 - U^iU_i \;=\; c^2, \tag{4.19}$$

with the Minkowski metric $\eta = \mathrm{diag}(1, -1, -1, -1)$. Also considering the 3-contraction of V^i and evaluating at $|v| = v_c$ gives

$$V^i V_i\Big|_{|v|=v_c} \;=\; v_0^2\left[\frac{2(\frac{v_c}{v_0})}{1-(\frac{v_c}{v_0})^2}\right]^2 \;=\; v_0^2\Big(\frac{mv_0^2 b}{k}\Big)^2 \;=\; v_0^2(\varepsilon^2-1), \tag{4.20}$$

where in the first step we used Eq. (4.3), the second step used Eq. (4.5), and the last step rewrote ζ in terms of ε in Eq. (4.14). As in Eq. (4.19), we can also write the 4-vector contraction of V^μ in Eq. (4.2) as

$$\eta_{\mu\nu}V^\mu V^\nu \;=\; V^0 V_0 - V^i V_i \;=\; v_0^2. \tag{4.21}$$

Comparing Eq. (4.18) and (4.20), we verify that the 3-velocities U^i and V^i are also equal up to orthogonal transformations. Thus, we have verified the equivalence of 4-velocities U^μ and V_μ under the duality. Indeed, the similarity in the expression of ε and b in Eq. (4.4) and (4.5) to the stereographic map to V^μ in Eq. (4.3) was no coincidence.

Given the 4-velocities, it is easy to obtain the 4-momenta. In special relativity, the 4-momentum P_μ is simply

$$P_\mu = m\eta_{\mu\nu}U^\nu, \tag{4.22}$$

where m is the rest mass of a relativistic particle. Its 4-vector contraction yields

$$\eta^{\mu\nu}P_\mu P_\nu = m^2, \tag{4.23}$$

where we set $c = 1$. This is the familiar energy-momentum relation. Following the analogy, the 4-momentum X^μ in the scattering-state Kepler problem is

$$X_\mu = m\eta_{\mu\nu}V^\nu, \tag{4.24}$$

where m is the mass of the non-relativistic scattered particle. Similarly, the 4-vector contraction yields

$$\eta^{\mu\nu}X_\mu X_\nu = m^2, \tag{4.25}$$

where we set $v_0 = 1$. Since we already verified the duality of 4-velocities, the duality of 4-momenta P_μ and X_μ followed trivially by a lowering of indices and a rescaling by m.

We now argue for the equivalence of 4-positions under the duality. First considering relativity, recall that we are only considering the set of all inertial frames within the null cone sharing the spacetime origin. That is, we are not considering translations. Then 4-displacement becomes the 4-position Q^μ, and its invariance relation is

$$\eta_{\mu\nu}Q^\mu Q^\nu = s^2, \tag{4.26}$$

where s is the "distance" between the origin and a spacetime event in the Minkowski metric. The quantity s can also be interpreted as the proper time τ measured by an inertial-frame observer traveling from the origin to an event. The invariance relation in Eq. (4.26) defines H^3, which is the set of all points of equal Minkowski distance s from the origin. It is known that the cotangent bundle of flat Minkowski spacetime is the 8-dimensional phase space $T^*\mathbb{R}^{1,3} = \mathbb{R}^{1,3} \oplus \mathbb{R}^{1,3}$, which includes spacetime and energy-momentum. We are then free to consider the canonical exchange of Q^μ and P_μ, meaning that the cotangent bundle of the energy-momentum space is equivalent to the cotangent bundle of spacetime. In relativity, particles of rest mass m are constrained on the invariant mass shell H^3 in Eq. (4.23). Exchanging Q^μ and P_μ, this is just a problem of a free particle on H^3 embedded in $\mathbb{R}^{1,3}$. We have already seen the treatment of this problem in the Moser Method of Chapter 3, where we obtained the coordinates X^μ and Y^μ of $\mathbb{R}^8$ for the scattering-state Kepler problem. If the Kepler-Lorentz duality holds for the cotangent bundles, not just the momentum manifolds, we should be able to show the invariance of H^3 of the cotangent bundle, i.e. $\eta_{\mu\nu}Y^\mu Y^\nu =$ constant, as in the Moser method. Recall from Chapter 3 that

$$Y_0 = -x^k y_k, \qquad Y^i = \frac{1}{2}\big(x^k x_k - 1\big)y_i - \big(x^k y_k\big)x^i, \tag{4.27}$$

where we used the index notation. Contracting Y^μ with itself in the Minkowski metric and simplifying, we get

$$\eta_{\mu\nu}Y^\mu Y^\nu = -\frac{1}{4}\Big(x^k x_k - 1\Big)^2\big(y^k y_k\big) = 2F, \tag{4.28}$$

where F is the Hamiltonian defined in Chapter 3. Recall that in order to retrieve the Kepler Hamiltonian H, we fixed the $F = \frac{1}{2}$ submanifold of $T^*\mathbb{R}^3$ for the scattering case, which corresponding to geodesic motion on a constant energy surface. Therefore, the same invariance relation holds. The coordinate space curves of the scattering state Kepler problem is given by a parallel projection of geodesics on this H^3 [5]. If we compare Eqs. (4.26) and (4.28), we see that the invariance of the spacetime distance s in relativity corresponds to the conservation of energy $\sqrt{2F}$ in the Kepler problem. Note that while the v_s gave us the duality of transformations, v_c gave the duality relations of objects subject to the transformations.

4.3.3 Boost Transformations via SL(2,C)

There is another, perhaps more elegant, formulation of special relativity in terms of actions on **spinors** instead of 4-vectors [13, 17]. We know that the representations of the Lorentz group $\Lambda^{\mu}{}_{\nu} \in SO(1,3)$ act on 4-vectors X^μ like

$$X^\mu \to X'^\mu = \Lambda^{\mu}{}_{\nu} X^\nu$$

as seen in the previous section. Let us consider an isomorphism that takes 4-vectors to Hermitian matrices:

$$X^\mu = \begin{pmatrix} X^0 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix} \quad \mapsto \quad \mathbf{X} = X^0 \mathbb{1} + X^i \boldsymbol{\sigma}_i = \begin{pmatrix} X^0 + X^3 & X^2 - iX^3 \\ X^2 + iX^3 & X^0 - X^3 \end{pmatrix},$$

where $\mathbb{1}$ is the 2×2 identity matrix and $\boldsymbol{\sigma}_i$ are Pauli matrices. The matrix $\mathbf{X}$ takes most general form of Hermitian matrix, and its determinant gives the usual invariance condition

$$\det \mathbf{X} = (X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2.$$

Which group, then, acts on $\mathbf{X}$ to induce transformations that amount to Lorentz transformations? There is a double cover of the Lorentz group called the Möbius group, denoted $SL(2,\mathbb{C})$. Representations of $SL(2,\mathbb{C})$ can generally be written

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}), \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{C}, \tag{4.29}$$

and $\mathbf{X}$ transforms under $SL(2,\mathbb{C})$ like

$$\mathbf{X} \to \mathbf{X}' = \mathbf{A}\mathbf{X}\mathbf{A}^\dagger, \tag{4.30}$$

where the dagger $\dagger$ is the Hermitian adjoint operation. This amounts to the Lorentz transformation of $\mathbf{X}$. If $\det \mathbf{X} = 0$, it is possible to decompose $\mathbf{X}$ into the outer product of two identical spinors $\xi \in \mathbb{C}^2$ such that $\mathbf{X} = \xi\xi^\dagger$. In this sense, a spinor is a "square root" of a 4-vector. Note that

$$\mathbf{X}' = \mathbf{A}\mathbf{X}\mathbf{A}^\dagger = \mathbf{A}\xi\xi^\dagger\mathbf{A}^\dagger = (\mathbf{A}\xi)(\mathbf{A}\xi)^\dagger,$$

so the spinor ξ transforms like

$$\xi \to \xi' = \mathbf{A}\xi$$

under the action of $SL(2,\mathbb{C})$. Though such spinors are fascinating, in our case $\det \mathbf{X} \neq 0$, meaning that the 4-vector X^μ isomorphic to $\mathbf{X}$ is not null. Nevertheless, our short discussion of spinors here will aid the understanding of twistors.

Let us turn our attention to the representations of $SL(2,\mathbb{C})$, which turns out to be a more natural setting for the Kepler-Lorentz duality than are than representations of $SO(1,3)$. An interesting property of $SL(2,\mathbb{C})$ is that the angles of generalized rotations (boosts and rotations) are halved. For example, spinors are carried to the negatives of themselves under a pure rotation by 2π and only comes back to itself under a pure rotation by 4π. Representations of $SL(2,\mathbb{C})$ inducing pure boosts can be written generally as

$$\mathbf{A} = \pm \exp\Big(-\frac{\phi}{2} n^i \boldsymbol{\sigma}_i\Big), \tag{4.31}$$

where n^i is a unit vector specifying the direction of the boost. In matrix form, this can be decomposed as into $\cosh\frac{\phi}{2}$ and $\sinh\frac{\phi}{2}$. Recall the expression of the quantity

$$\frac{v_c}{v_0} = \tanh\frac{\phi}{2}, \tag{4.32}$$

the velocity of particle at the point of closest approach to the force center, which we obtained via the metric on $\mathcal{P}$. Then, the boosts of $SL(2,\mathbb{C})$ can be expressed in terms of the quantities

$$\begin{aligned} \cosh\frac{\phi}{2} &= \frac{1}{\sqrt{1-(\frac{v_c}{v_0})^2}}, \\ \sinh\frac{\phi}{2} &= \frac{(\frac{v_c}{v_0})}{\sqrt{1-(\frac{v_c}{v_0})^2}}. \end{aligned} \tag{4.33}$$

Hence, the new scattering variables naturally give us representations of $SO(1,3)$ as well as $SL(2,\mathbb{C})$. Curiously, the boost representations of $SO(1,3)$ in terms of v_s and those of $SL(2,\mathbb{C})$ in terms of v_c take the same mathematical form. The new variables make physically and geometrically explicit the relationship between the transformations of 4-vector and those of rank-2 spinors.

How are v_c and v_s related geometrically? While v_c is a quantity inside the Poincaré ball $\mathcal{P}$, v_s is a quantity at the conformal boundary of $\mathcal{P}$ (Fig. 4.1). By the duality, we can easily obtain the quantity $\bar{v}$, dual to v_c, inside $\mathcal{P}$ of relativity by Eq. (4.8):

$$\frac{\bar{v}}{c} = \frac{(\frac{v}{c})}{1+\sqrt{1-(\frac{v}{c})^2}}. \tag{4.34}$$

Our analysis for the spinor formulation of the scattering-state Kepler problem suggests that $\bar{v}$ may be a natural quantity for boost representations in the spinor formulation of relativity. It would thus be of interest to see whether such a quantity turns out to be of use in the spinor formulation of relativity. The ideas presented in this chapter are simple, yet subtle. More discussion along the lines of the geometry noted here is given in Chapter 5.

Chapter 5

Conformal Symmetry and Twistor Theory

The Kepler-Lorentz duality, presented in Chapter 4, provides an elegant framework in which to think about the dynamics of the scattering Kepler problem and the kinematics of flat spacetime as different physical manifestations of the same mathematical problem. We would ideally like to use this classical duality as a starting point to gain new theoretical insights into both systems. However, as the Kepler problem and special relativity are both thoroughly investigated topics in physics, it can be difficult to immediately think of a way to extract more theoretical insights from the duality.[1]

In this chapter, we seek to uplift the Kepler problem and special relativity to the less explored conformal symmetric context, where the duality may possibly be used to shed light on each other. The advantages of doing so in the Kepler problem are many. By moving to a conformal invariant manifold, we get a unified picture of the problem for both positive and negative energies, obtained by taking various cross sections of the manifold. In doing so, we retrieve the Moser map of the phase manifold, which can be used to retrieve the Kepler Hamiltonian and other integrals of motion. We then cast the problem in the context of twistor theory [18], originally proposed by Sir Roger Penrose, where we retrieve the Kustaanheimo-Stiefel map between the 3-dimensional scattering (positive energy) Kepler problem and the 4-dimensional inverted harmonic oscillator. Together with the Kepler-Lorentz duality, this suggests the equivalence of the 3D scattering state Kepler problem, 4D relativistic free-particle kinematics, and 4D inverted harmonic oscillator.

Although we will be using concepts from advanced mathematics,[2] we avoid the

[1]Besides the spinor formulation noted in Chapter 4, an exception may be the use of the famous four-vector techniques from special relativity to facilitate some computations in the scattering Kepler problem. However, this example still sees the duality as a computational tool rather than as an approach to gain new theoretical insights.

[2]Namely, Lie groups and algebras, differential geometry, and Hamiltonian mechanics in the symplectic formulation. To gain more background on these topics, consult [4, 13, 19–21] and the appendices of [2]

rigorous theorem-proof style to emphasize the physical picture. We instead choose to explain any new mathematical concepts introduced in more intuitive terms to suggest a way in which one can think about the concept. Indeed, the aim of this chapter is to paint possible directions of the research in broad strokes. This chapter follows Ch. 6-7 of Cordani [2] closely regarding the mathematical treatment of the Kepler problem, and readers interested in the proofs of some of the claims made in this chapter may refer there. However, the insights gained through the Kepler-Lorentz duality endows the results in the Kepler problem an alternative side of physical interpretations.

We use the index and inner product conventions used in [2]. The range of the indices are

$$A, B, C = -1, 0, 1, \cdots, 4$$
$$\mu, \nu, \rho, \sigma = 0, 1, 2, 3$$
$$i, j, k = 1, 2, 3$$

We use $(\cdot, \cdot) : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ for the 3-dimensional scalar product and $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ for the pairing between a Lie algebra $\mathfrak{g}$ and its dual algebra $\mathfrak{g}^*$. For convenience, we also write $x = \sqrt{(x, x)}$ and $y = \sqrt{(y, y)}$. Otherwise, we use the Einstein index notation.

5.1 The Conformal Group

Let us define the flat Minkowski space M_0 as $\mathbb{R}^{1,3} = \mathbb{R}^1 \oplus \mathbb{R}^3$ equipped with the pseudo-Euclidean metric tensor $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. The identity connected component of the 6-dimensional Lorentz group $\mathfrak{L}_0 = SO_0(1, 3)$, together with the 4-dimensional translation group $\mathfrak{T}$ in M_0, form the 10-dimensional **Poincaré group** $\mathfrak{P} = \mathfrak{L}_0 \times_S \mathfrak{T}$ via a semi-direct product.[3] The action of the Poincaré group $\mathfrak{P}$, exhausting all possible "rigid motions" on M_0, preserves the metric $\eta_{\mu\nu}$. Transformations preserving angles between any two lines in $\mathbb{R}^{1,3}$ are called conformal maps, and they are elements of the **conformal group** $C(1, 3)$ of Minkowski space M_0. An equivalent way of defining the conformal group $C(1, 3)$ of M_0 is the group of isomorphisms of smooth manifolds, or "diffeomorphisms", that transform the metric like

$$\eta_{\mu\nu} \mapsto g_{\mu\nu}(x) = w^2(x)\eta_{\mu\nu},$$

where $x \in M_0$ and $w(x)$ is a non-vanishing function. Manifolds with the metric tensor $w^2(x)\eta_{\mu\nu}$ are **conformally related** to the Minkowski space M_0, meaning that there exists an element of $C(1, 3)$ that takes such a manifold to a manifold with the Minkowski metric $\eta_{\mu\nu}$. Choosing $w(x) = 1$, we see that the Poincaré group $\mathfrak{P}$ is a subgroup of

[3] Here, the semi-direct product instead of the direct product must be used since $\mathfrak{L}_0$ and $\mathfrak{T}$ are non-compact groups.

$C(1,3)$. To look for all local nontrivial conformal maps, let us consider the most general form of the infinitesimal conformal transformation

$$x^\mu \mapsto \bar{x}^\mu = x^\mu + \epsilon V^\mu(x) + O(\epsilon^2)$$

satisfying

$$\frac{\partial \bar{x}^\mu}{\partial x^\rho}\frac{\partial \bar{x}^\nu}{\partial x^\sigma}\eta_{\mu\nu} = w^2(x)\eta_{\rho\sigma}, \quad \text{with } w(x) = 1 + \epsilon f(x) + O(\epsilon^2),$$

where the terms on the left hand side of the equation above are simply the transformation Jacobians from x^μ to $\bar{x}^\mu$. We already know that the zeroth order term in ϵ should yield the trivial solutions corresponding to the Poincaré group $\mathfrak{P}$, so we keep only the first order terms in ϵ. Comparing the first order terms in ϵ, we obtain the flat space conformal Killing Equation

$$\partial_\mu V_\nu + \partial_\nu V_\mu = 2f(x)\eta_{\mu\nu},$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$. It is known that the general solution of the conformal Killing Equation is given by

$$V_\mu(x) = t_\mu + \lambda_{\mu\nu}x^\nu + dx_\mu + 2k_\nu x^\nu x_\mu - k_\mu x_\nu x^\nu, \tag{5.1}$$

where $\lambda_{\mu\nu} = -\lambda_{\nu\mu}$ and the solution depends on 15 parameters t_μ, $\lambda_{\mu\nu}$, d, and k_μ. This is the conformal vector field.

In differential geometry, the vector field gives a tangent vector at each point on the manifold [4, 13, 21]. This relation in the transformed coordinates is

$$\frac{d\bar{x}^\mu}{d\tau} = V^\mu(\bar{x}),$$

where τ is the parameter along a curve on the manifold from which the tangent vector is defined. Integrating while keeping only one of the four classes of parameters nonzero, we get the following four transformations of $C(1,3)$:

i For $t^\mu \neq 0$, we get $\bar{x}^\mu = x^\mu + t^\mu\tau$, the translation group $\mathfrak{T}$,

ii For $\lambda^\mu{}_\nu \neq 0$, we get $\bar{x}^\mu = \Lambda^\mu{}_\nu(\tau)x^\nu$ with $\boldsymbol{\Lambda}(\tau) = \exp(\boldsymbol{\lambda}\tau)$, the Lorentz group $\mathfrak{L}_0$,

iii For $d \neq 0$, we get $\bar{x}^\mu = \exp(d\tau)x^\mu$, the dilation group $\mathfrak{D}$,

iv For $k^\mu \neq 0$, we get

$$\bar{x}^\mu = \frac{x^\mu - \tau k^\mu x^\nu x_\nu}{1 - 2\tau k_\nu x^\nu + \tau^2 k^\nu k_\nu x^\alpha x_\alpha},$$

the special conformal transformation group $\mathfrak{K}$.

While other transformations are transparent in their physical interpretation, the meaning of special conformal transformations are unclear. They can be understood as the composition of the operations invert-translate-invert:

$$x^\mu \to \frac{x^\mu}{x^\nu x_\nu} \to \frac{x^\mu}{x^\nu x_\nu} - \tau k^\mu \to \frac{x^\mu - \tau k^\mu x^\nu x_\nu}{1 - 2\tau k_\nu x^\nu + \tau^2 k^\nu k_\nu x^\alpha x_\alpha}.$$

On the Poincaré ball $\mathcal{P}$ projected from H^3 embedded in Minkowski space M_0, special conformal transformations physically amount to turning the repulsive Kepler problem inside $\mathcal{P}$ to an attractive problem outside $\mathcal{P}$ by taking $k \to -k$ of the coupling constant, translating in stereographic coordinates, then taking again $k \to -k$ to a repulsive problem inside $\mathcal{P}$. It is also immediately seen that the the special conformal transformations become singular at a finite value of τ. The transformation can be regularized by adding a null cone at infinity to the Minkowski space M_0 to form the compactified Minkowski space M discussed in the next section.

Let us express the 15 generators of the Lie algebras, corresponding to each of the four subgroups of $C(1,3)$ above, in the basis $\{\partial_\mu\}$ of the conformal vector field. The generators serve as the basis spanning the Lie algebra, which can then be exponentiated to form elements of its corresponding Lie group. Thus, the conformal group $C(1,3)$ is formed by a 15-dimensional conformal Lie algebra $c(1,3)$.

i $\mathcal{P}_\mu = \partial_\mu$, of the Lie algebra $\mathfrak{t}$ of the translation group $\mathfrak{T}$,

ii $\mathcal{G}_{\mu\nu} = x_\nu \partial_\mu - x_\mu \partial_\nu$, of the Lie algebra $\mathfrak{l}_0$ of the Lorentz group $\mathfrak{L}_0$,

iii $\mathcal{D} = x^\mu \partial_\mu$, of the Lie algebra $\mathfrak{d}$ of the dilation group $\mathfrak{D}$,

iv $\mathcal{K}_\mu = 2x_\mu x^\nu \partial_\nu - x^\nu x_\nu \partial_\mu$, of the Lie algebra $\mathfrak{k}$ of the special conformal transformation group $\mathfrak{K}$.

It is tedious, but not difficult, to compute their Lie bracket relations, defined $[A, B] = AB - BA$:

$$\begin{gathered}
[\mathcal{G}_{\alpha\mu}, \mathcal{G}_{\alpha\nu}] = \eta_{\alpha\alpha}\mathcal{G}_{\mu\nu} \text{ or } = 0 \text{ if all indices different} \\
[\mathcal{G}_{\mu\nu}, \mathcal{P}_\alpha] = \eta_{\mu\alpha}\mathcal{P}_\nu - \eta_{\nu\alpha}\mathcal{P}_\mu \\
[\mathcal{G}_{\mu\nu}, \mathcal{K}_\alpha] = \eta_{\mu\alpha}\mathcal{K}_\nu - \eta_{\nu\alpha}\mathcal{K}_\mu \\
[\mathcal{P}_\mu, \mathcal{K}_\nu] = 2\mathcal{G}_{\mu\nu} + 2\mathcal{D}\eta_{\mu\nu} \\
[\mathcal{P}_\mu, \mathcal{D}] = \mathcal{P}_\mu \\
[\mathcal{K}_\mu, \mathcal{D}] = -\mathcal{K}_\mu \\
[\mathcal{G}_{\mu\nu}, \mathcal{D}] = [\mathcal{P}_\mu, \mathcal{P}_\nu] = [\mathcal{K}_\mu, \mathcal{K}_\nu] = 0.
\end{gathered}$$

The Lie bracket relations for $c(1,3)$ suggest a nonlinear Lie group structure, which is a disappointment. However, let us consider the generators $\mathcal{G}_{AB}$ formed by linear combinations of the above generators:

$$\mathcal{G}_{\mu\nu}, \quad \mathcal{G}_{\mu,4} = \frac{1}{2}(\mathcal{P}_\mu + \mathcal{K}_\mu), \quad \mathcal{G}_{-1,\mu} = \frac{1}{2}(\mathcal{P}_\mu - \mathcal{K}_\mu), \quad \mathcal{G}_{-1,4} = \mathcal{D},$$

where the $\mathcal{G}_{\mu\nu}$ of $\mathcal{G}_{AB}$ remain generators of Lorentz algebra $\mathfrak{l}_0$. Remarkably, the Lie bracket relations simplify to the following:

$$[\mathcal{G}_{AB}, \mathcal{G}_{AC}] = \eta_{AA}\mathcal{G}_{BC} \text{ or } = 0 \text{ if all indices different,} \tag{5.2}$$

where $\eta_{AB} = \mathrm{diag}(1, 1, -1, -1, -1, -1)$. The 15 generators $\mathcal{G}_{AB}$ form a Lie algebra isomorphic to that of the pseudo-orthogonal Lie group $SO(2,4)$! Let us consider $X^A \in \mathbb{R}^{2,4}$, where $SO(2,4)$ acts linearly on $\mathbb{R}^{2,4}$. Then the 15 generators/vector fields $\mathcal{G}_{AB}$ can be written as

$$\mathcal{G}_{AB} = X_B\partial_A - X_A\partial_B, \tag{5.3}$$

spanning the Lie algebra of $SO(2,4)$. Since the generators of the Lie algebras $c(1,3)$ and $so(2,4)$ are related only by the change of basis in the subspace spanned by $\mathcal{P}_\mu$ and $\mathcal{K}_\mu$, the groups $C(1,3)$ and $SO(2,4)$ are locally isomorphic. In fact, $SO(2,4)$ is globally a double cover of $C(1,3)$, i.e. $C(1,3) = SO(2,4)/\mathbb{Z}_2$. That is, the neighborhood of both I and $-I$ of $SO(2,4)$ look locally like the neighborhood of I of $C(1,3)$, where I is the identity element. Needless to say, it is more productive to consider the space $\mathbb{R}^{2,4}$ on which $SO(2,4)$ and $C(1,3)$ act linearly.

5.2 The Compactified Minkowski Space

As noted in the above, the compactification of Minkowski space regularizes the conformal transformations by adding a null cone at the infinity. The action of $SO(2,4)$ on $\mathbb{R}^{2,4}$ preserves the null cone $\mathcal{N}$, defined as

$$\mathcal{N} = \{X \in \mathbb{R}^{2,4} : \eta(X,X) = 0\}.$$

Let us define M as the manifold of null unoriented rays through the origin of $\mathbb{R}^{2,4}$. The action of $SO(2,4)$ on M is transitive; that is, there exists an element of $SO(2,4)$ that carries a point $m \in M$ to any other point $m' \in M$ for all m. But how does this relate to adding a null cone at the infinity? Let us split $\mathbb{R}^{2,4}$ as follows: $\mathbb{R}^{2,4} = \mathbb{R}^{1,1} \oplus \mathbb{R}^{1,3}$. We can also split $X \in \mathbb{R}^{2,4}$ into $X = X_{(1,1)} + X_{(1,3)}$, where $X_{(1,1)} \in \mathbb{R}^{1,1}$ of indices -1,4 and $X_{(1,3)} \in \mathbb{R}^{1,3}$ of indices $0, 1, 2, 3$. If X belongs to the null cone N, i.e. $\eta(X,X) = 0$, then

$$\eta(X_{(1,3)}, X_{(1,3)}) = -\eta(X_{(1,1)}, X_{(1,1)}) = \text{constant},$$

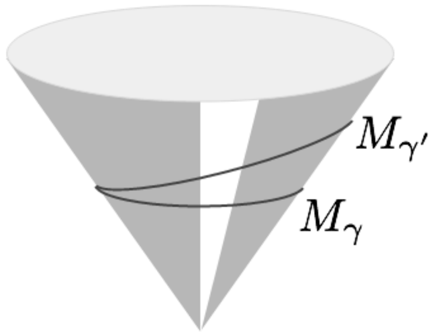

FIGURE 5.1: Null cone of $\mathbb{R}^{2,4}$ and its conformally-related cross-sections

where the norms yield the equations for hyperboloids H^3 and H^1 for nonzero constant. The same unoriented ray is associated with both X and $-X$, so this gives us a manifold M homeomorphic to $(H^1 \times H^3)/\mathbb{Z}_2$. The H^3 is an invariant manifold under the action $SO(1,3)$ on Minkowski space $M_0 = \mathbb{R}^{1,3}$, so the presence of H^1 with negative of the norm of H^3 forms the compactified Minkowski space M. If we take $\eta(X_{(1,3)}, X_{(1,3)}) = 0$ and $\eta(X_{(1,1)}, X_{(1,1)}) = 0$, the compactified Minkowski space M can be interpreted as the addition of a null cone at the infinity of the Minkowski null cone. It is also easy to show that M is homeomorphic to $(S^1 \times S^3)/\mathbb{Z}_2$ by similar means, to demonstrate that M is indeed compact.[4]

The compactification is nice from a mathematical perspective, but we would like to retrieve the physical phase manifolds for various energies seen in the Moser method of Ch. 2. To do this, define $\widetilde{M}$ as the manifold of oriented null lines through the origin of $\mathbb{R}^{2,4}$. Equipped with an orientation, it is a double cover of M, i.e. $M = \widetilde{M}/\mathbb{Z}_2$. Let us choose a 5-dimensional submanifold of $\mathbb{R}^{2,4}$ intersecting the 5-dimensional null cone $\mathcal{N}$ at most once. Note that $\dim \widetilde{M} = 4$, obtained by subtracting the dimension of the null rays themselves from $\dim \mathcal{N}$. The intersection gives us a map $\gamma : \widetilde{M} \to M_\gamma$, where M_γ is the cross-section between the submanifold and $\mathcal{N}$. It also holds that $\dim M_\gamma = 4$, suggesting its description in terms of the coordinates on the manifold $H^1 \times H^3$, where H^3 is the manifold seen in the scattering case of the Moser method. We will explicitly calculate the expression of M_γ for various energies in the subsequent sections. It is possible to prove that the cross-sections M_γ are all conformally related to the Minkowski space M_0, such that the metric tensor on M_γ only differs from the Minkowski metric $\eta_{\mu\nu}$ by a non-vanishing function $w^2(x)$ for all $x \in M_\gamma$. Thus, $\widetilde{M}$ and M are endowed with a whole class of metrics whose elements are all conformally related. Two conformally related cross sections M_γ and $M_{\gamma'}$ are depicted in Fig. 5.1.

[4]Typically, it is shown in the literature that M is homeomorphic to $(S^1 \times S^3)/\mathbb{Z}_2$, We have shown the homeomorphism to $(H^1 \times H^3)/\mathbb{Z}_2$ instead for the following reasons: (1) we seek to make the global topology of the compactified Minkowski space M more intuitive in relation to the Minkowski space M_0 and (2) we are primarily concerned with the positive energy case, where we have established the duality.

While it is $SO(2,4)$ that induces the automorphism (self-map) of $\widetilde{M}$, the mod $\mathbb{Z}_2$ in the double cover relation $C(1,3) = SO(2,4)/\mathbb{Z}_2$ necessitates the irrelevance of the orientation of the null rays on the manifold on which $C(1,3)$ acts. Thus, we can take $C(1,3)$ to induce the automorphism of M. The following commutative diagram illustrates the relationships of objects presented in this section:

$$\begin{tikzcd}
M_\gamma \arrow[r, "SO(2,4)"] & M_\gamma \\
\widetilde{M} \arrow[u, "\gamma"] \arrow[r, "SO(2,4)"] \arrow[d, "\pi"'] & \widetilde{M} \arrow[u, "\gamma'"'] \arrow[d, "\pi"] \\
M \arrow[r, "C(1,3)"] & M
\end{tikzcd}$$

where $\pi : \tilde{M} \to M$ is a 2-1 projection from oriented to unoriented null lines.

5.3 The Moment Map

The action of Lie group $\mathfrak{G}$ on the phase manifold T^*M induces a map from its Lie algebra $\mathfrak{g}$ to a vector field on T^*M. That is, $L \mapsto \xi_L$ where L is an element of $\mathfrak{g}$ and ξ_L is the vector field corresponding to L. This is seen in Eq. (5.4), where the words "generators" and "vector fields" have been used interchangeably. On the other hand, a smooth function $f \in C^\infty(T^*M)$[5] induces a vector field on the cotangent bundle T^*M through the following expression introduced in Ch. 3:

$$\xi_f = \{\cdot, f\} = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}, \tag{5.4}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket. If $\xi_L = \xi_f$, it is useful to define the map that takes an element $L \in \mathfrak{g}$ to the corresponding smooth function $\mu_L = f \in C^\infty(T^*M)$. The map that induces this is called the **moment map** [19, 20]. The moment map acts on the symmetry generator L to give the function μ_L, and can thus be understood as the Hamiltonian (and arguably a more elegant) version of the Noether theorem.[6]

Let us first present some definitions which will be useful in defining the moment map. A **bivector** of a (pseudo-)Riemannian manifold is an antisymmetric tensor of

[5] $C^\infty(T^*M)$ is the space of smooth functions on T^*M.

[6] Notice that the moment map is not an isomorphism between L and $f = \mu_L$, since $f + C$ where C is a constant will give the same vector field ξ_f due to the derivative in the Poisson bracket.

order 2 (2-form). If a bivector can be expressed as the exterior (wedge) product of two vectors Y and X, it is called a simple bivector $Y \wedge X$. In component form, a simple bivector can be expressed as $(Y \wedge X)_{AB} = Y_A X_B - Y_B X_A$. Let us consider the manifold $\mathcal{S}$ of null simple bivectors G of $\mathbb{R}^{2,4}$ defined

$$\mathcal{S} = \{G = Y \wedge X : 0 \neq X, Y \in \mathbb{R}^{2,4}, \eta(G,G) = 0\}.$$

Since we restricted ourselves to $X, Y \neq 0$ above, we will hereafter consider T^+M, the cotangent bundle T^*M minus the zero section, instead of T^*M. Let us find $\dim \mathcal{S}$. The vectors X and Y together contribute $\dim \mathbb{R}^{2,4} \oplus \mathbb{R}^{2,4} = 12$ dimensions. The nullity condition

$$\eta(X,X)\eta(Y,Y) - \eta^2(X,Y) = 0.$$

provides 1 constraint. The map $X, Y \mapsto X \wedge Y$ is not one-to-one, since any set of vectors differing from X, Y by the action of the group $SL(2,\mathbb{R})$ also maps to $X \wedge Y$. That is, given vectors X', Y' such that

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{R},$$

it holds that $X' \wedge Y' = X \wedge Y$. The determinant condition allows matrix representations of $SL(2,\mathbb{R})$ to be specified by 3 real parameters, providing 3 more constraints. Hence,

$$\dim \mathcal{S} = 12 - 1 - 3 = 8,$$

which is the number of dimensions expected for the cotangent bundle T^+M, since $\dim M = 4$. Indeed, we can show that $\mathcal{S}$ is homeomorphic to T^+M, but we will not prove this here. The vectors $X, Y \in \mathcal{S}$ can now be interpreted as coordinates on the phase manifold T^+M, and thus $\eta(X,X) = \eta(X,Y) = 0$.

Given the phase manifold T^+M with symmetry group $\mathfrak{G}$ such that $\mathfrak{G} : T^+M \to T^+M$, we can define the moment map

$$\mu : T^+M \to \mathfrak{g}^*,$$

where $\mathfrak{g}^*$ is the dual algebra of the Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$. With $\mathfrak{G} = C(1,3)$, the explicit form of the moment map is

$$\mu = \sum_{A<B} G_{AB} \mathcal{G}^*_{AB} = \sum_{A<B} (Y_A X_B - Y_B X_A) \mathcal{G}^*_{AB}, \tag{5.5}$$

where G is the bivector of coordinates on T^+M such that $\eta(G,G) = 0$, and $\mathcal{G}^*_{AB}$ are the dual generators of $c^*(1,3)$.[7] We can act $\mathcal{G}^*_{AB}$ on the corresponding generators $\mathcal{G}_{AB}$ of $c(1,3)$ such that the pairing gives $\langle \mathcal{G}^*_{AB}, \mathcal{G}_{A'B'} \rangle = 1$ for identical indices $AB = A'B'$. The desired map $\mu_L : \mathfrak{g} \to C^\infty(T^+M)$ for $L \in c(1,3)$ is defined as follows:

$$\mu_L = \langle \mu, L \rangle. \tag{5.6}$$

We send $L \in \mathfrak{g}$ to its corresponding smooth function $\mu_L \in C^\infty(T^+M)$, which can be decomposed as a linear combination of the bivector components $G_{AB} = Y_A X_B - Y_B X_A$. Since T^+M is symmetric under $C(1,3)$, μ_L are conserved functions on T^+M corresponding to the symmetry generated by L. Note that if we only choose elements L of the Lie algebra $c(1,3)$ that coincide with the generators $\mathcal{G}_{AB}$ of $c(1,3)$, we get $\mu_L = G_{AB}$ for $L = \mathcal{G}_{AB}$. Hence, when we act the moment map μ on $\mathcal{G}_{AB}$, we are projecting out the conserved quantities G_{AB} of Eq. (5.5).

In Chapter 2, we claimed that smooth functions $f = \mu_L$ on T^+M are the generators of the Lie algebra $\mathfrak{g}$. In this section, we have made this statement more rigorous via the action of the moment map $L \mapsto \mu_L$, which reveals the relationship between the the element of $\mathfrak{g}$ generating a symmetry and its corresponding integral of motion. Due to this correspondence, which is unique up to an additive constant on the function, the Poisson algebra of the conserved quantities,

$$\{G_{AB}, G_{AC}\} = \eta_{AA} G_{BC} \text{ or } = 0 \text{ if all indices different,}$$

is isomorphic to the Lie algebra of the conformal group,

$$[\mathcal{G}_{AB}, \mathcal{G}_{AC}] = \eta_{AA} \mathcal{G}_{BC} \text{ or } = 0 \text{ if all indices different.}$$

5.4 Connection to Moser Method

The Moser method, presented in Chapter 3, demonstrates the equivalence of the Kepler problem to the geodesic flow on a 3-manifold. The geodesic flow gave us a more natural Hamiltonian J that regularizes the Kepler problem, and we derived the canonical transformation between the cotangent bundle of the 3-manifold and the phase space of the Kepler problem. We show in this section that the Hamiltonian J and the canonical transformations can be elegantly derived using the moment map. The negative and positive energy cases are treated in a unified picture, where they are simply different cross-sections M_γ of the null cone $\mathcal{N} \in \mathbb{R}^{2,4}$. Remarkably, the conformal symmetry gives

[7] Although $SO(2,4)$ is globally a double cover of $C(1,3)$, it holds that $so(2,4) = c(1,3)$, since Lie algebras are defined locally in the neighborhood of the identity element. Therefore, $\mathcal{G}^*_{AB}$ are generators of both $so^*(2,4)$ and $c^*(1,3)$.

an extended phase manifold of the Kepler problem, where some group-geometric properties of the Kepler-Lorentz duality seem to be in common with those of gauge-gravity duality.

5.4.1 The Pullback

To retrieve the cotangent bundle of the 3-manifold on which the Kepler problem is equivalent to the geodesic flow, we consider different cross sections M_γ of the null cone $\mathcal{N} \in \mathbb{R}^{2,4}$. The 4-dimensional submanifolds M_γ have metric tensors that are related to the ambient metric $\eta_{AB} = \text{diag}(1, 1, -1, -1, -1, -1)$ via

$$(g_\gamma)_{\mu\nu} = \frac{\partial X^A}{\partial x^\mu} \frac{\partial X^B}{\partial x^\nu} \eta_{AB}, \tag{5.7}$$

where x^μ,y_μ are canonical coordinates on T^+M_γ and $X^A = X^A(x^\mu) \in \mathcal{N}$ are vectors on the section M_γ such that $\eta(X, X) = 0$. We still need to find the covectors $Y_A = Y_A(x^\mu, y_\nu)$ on the cotangent bundle. From differential geometry, we are guaranteed the map $\gamma^* : T^+M_\gamma \to T^+\mathcal{N}$ taking $y_\mu \mapsto Y_A$ if we have the map $\gamma : \mathcal{N} \to M_\gamma$ taking $X^A \mapsto x^\mu$ [13, 21].[8] The map γ^* is called the **pullback**. In words, the pullback maps a covector on the cotangent bundle of the target manifold back to a covector on the cotangent bundle of the initial manifold, given that we have a map from the initial manifold to the target manifold. The following commutative diagram illustrates the idea:

$$\begin{tikzcd} \mathcal{N} \arrow[r, "\gamma"] & M_\gamma \\ T^+\mathcal{N} \arrow[u, "\pi_{\mathcal{N}}"] & T^+M_\gamma \arrow[l, "\gamma^*"'] \arrow[u, "\pi_{M_\gamma}"'] \end{tikzcd}$$

where $\pi_{\mathcal{N}}$ and π_{M_γ} are the projection maps from the cotangent bundle to the base manifold. For the pullback, let us take

$$Y_B(x^\mu, y_\nu) = (g_\gamma)^{\mu\nu} \frac{\partial X^A}{\partial x^\mu} \eta_{AB} y_\nu, \tag{5.8}$$

which is indeed the simplest map consistent with the index notation that takes y_ν to Y_A.[9] Now that we have the construction for the covectors Y_A given γ, we can finally

[8] The pullback is actually defined at a cotangent space at a particular point of the base manifold. We ease the notation and make this implicit.

[9] One can also obtain the **push-forward** map $\gamma_* : T\mathcal{N} \to TM_\gamma$ of the tangent bundles taking $V^A \mapsto v^\mu$ where $V^A \in T\mathcal{N}$ and $v^\mu \in TM_\gamma$ and check that the projection constraint $V^A Y_A = v^\mu y_\mu$ is satisfied. This verifies Eq. (5.8) but we will not do this here.

write the function

$$G_{AB}(x^\mu, y_\nu) = Y_A X_B - Y_B X_A, \tag{5.9}$$

associated with the moment map $\mu : T^+ M_\gamma \to c^*(1,3)$ defined in Eq. (5.5).

With this, let us retrieve the Minkowski space M_0. Intersecting the null cone $\mathcal{N}$, given explicitly by

$$(X^{-1})^2 + (X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 - (X^4)^2 = 0, \tag{5.10}$$

with the hyperplane $X^{-1} + X^4 = 1$, we obtain the section

$$\begin{aligned} X^{-1} &= \frac{1}{2}(1 + \eta_{\mu\nu} x^\mu x^\nu) \\ X^0 &= x^0 \\ X^i &= x^k \\ X^4 &= \frac{1}{2}(1 - \eta_{\mu\nu} x^\mu x^\nu). \end{aligned}$$

The metric of the section, obtained via Eq. (5.7), is $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, which is the metric of the Minkowski space M_0. The pullback in Eq. (5.8) gives us

$$\begin{aligned} Y_{-1} &= -x^\mu y_\mu \\ Y_0 &= y_0 \\ Y_i &= y_k \\ Y_4 &= -x^\mu y_\mu. \end{aligned}$$

Then, we can find from Eq. (5.9) the moment map coefficients

$$\begin{aligned} G_{\mu\nu} &= x_\nu y_\mu - x_\mu y_\nu \\ P_\mu &= y_\mu \\ K_\mu &= 2x_\mu x^\nu y_\nu - x^\nu x_\nu y_\mu \\ D &= x^\mu y_\mu, \end{aligned} \tag{5.11}$$

where $P_\mu = G_{\mu,4}$, $K_\mu = G_{-1,\mu}$, and $D = G_{-1,4}$.[10] Comparing Eq. (5.11) with the generators of $c(1,3)$ in Sec. 5.1, we again confirm the correspondence between the Lie algebra generators and smooth functions on the cotangent bundle. Note, however, that under the Kepler-Lorentz duality, this space is not the Minkowski spacetime but rather the energy-momentum space with the Minkowski metric.

[10]This assignment of G_{AB} to functions P_μ and K_μ is different from the assignment seen for the generators of $so(2,4)$ in Sec. 5.1. In Sec. 5.1, we changed the basis to yield the simple Lie bracket relations in Eq. (5.2), but here we consider the G_{AB} before changing the basis.

5.4.2 Retrieving the Moser Method

Given the moment map and the pullback, it is now rather straightforward to obtain the cross sections for negative and positive energy Kepler problems [2]. In the negative energy (bound, attractive) case, we intersect the null cone $\mathcal{N}$ in Eq. (5.10) with the hypersphere

$$\sum_A (X^A)^2 = 2.$$

The cross section is given by

$$\begin{aligned} X^{-1} &= \cos x^0 \\ X^0 &= \sin x^0 \\ X^i &= \frac{2x_i}{x^2+1} \\ X^4 &= \frac{x^2-1}{x^2+1}, \end{aligned}$$

where we have expressed X^i, X^4 in the stereographic coordinates of S^3 seen in the Moser method. Note that the coordinates X^{-1}, X^0 form S^1. This confirms our claim in Sec. 5.2 that M is homeomorphic to $(S^1 \times S^3)/\mathbb{Z}_2$. The pullback and the metric of the section is

$$\begin{aligned} Y_{-1} &= -y_0 \sin x^0 \\ Y_0 &= y_0 \cos x^0 \\ Y_i &= \frac{1}{2}(x^2+1)y_i - (x,y)x_i \\ Y_4 &= (x,y) \end{aligned}$$

and

$$(g_-)^{\mu\nu} y_\mu y_\nu = y_0^2 - \left[\frac{1}{2}y(x^2+1)\right]^2.$$

The expressions of X^i, X^4 and Y_i, Y_4 correspond to that obtained in the Moser method in Chapter 3. The coordinates X^A, Y_A are parametrized by x^μ, y_μ, which spans 8 dimensions as required by $\dim \mathcal{S} = 8$. Also, the term in the square bracket of $(g_-)^{\mu\nu}$ is the metric on S^3 in stereographic coordinates.

In the positive energy (scattering, repulsive) case, we intersect the null cone $\mathcal{N}$ with the hyperboloid

$$(X^{-1})^2 - (X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 + (X^4)^2 = 1.$$

The cross section is given by

$$
\begin{aligned}
X^{-1} &= \frac{x^2+1}{x^2-1} \\
X^0 &= \sinh x^0 \\
X^i &= \frac{2x_i}{x^2-1} \\
X^4 &= \cosh x^0.
\end{aligned}
$$

We have expressed X^{-1}, X^i in the stereographic coordinates of H^3 seen in the Moser method. The coordinates X^0, X^4 indeed form H^1, and we confirm our claim in Sec. 5.2 that M is homeomorphic to $(H^1 \times H^3)/\mathbb{Z}_2$. The pullback and the metric gives

$$
\begin{aligned}
Y_{-1} &= (x, y) \\
Y_0 &= y_0 \cosh x^0 \\
Y_i &= \frac{1}{2}(x^2-1)y_i - (x,y)x_i \\
Y_4 &= -y_0 \sinh x^0
\end{aligned}
$$

and

$$
(g_+)^{\mu\nu} y_\mu y_\nu = y_0^2 - \left[\frac{1}{2} y(x^2-1)\right]^2,
$$

as in the Moser method. The term in the square bracket of $(g_+)^{\mu\nu}$ is the metric on H^3 in stereographic coordinates. Though we have not treated the attractive scattering case, its results are easy to obtain from the repulsive scattering problem. Recall that the repulsive problem occurs inside the Poincaré ball $\mathcal{P}$ while the attractive problem occurs outside $\mathcal{P}$. Thus, the results for the attractive case is found simply by inverting the stereographic coordinate, i.e. $x^i \mapsto \frac{x^i}{x^2}$.

Notice that in the metric, the term in square brackets is also related to the new Hamiltonians $J_{(-)}$ and $J_{(+)}$ of the Moser method for bound and scattering states. By the metric relation, y_0 is also related to the Hamiltonians and thus the energy. Hence, not only do we retrieve the Moser method involving the coordinates x^i, y_i spanning 6 dimensions but we also naturally get an enlarged phase manifold, where we see quantities related to energy and time, y_0 and x^0, treated on an equal footing with coordinates and conjugate momenta, x^i and y_i. This is why we have an enlarged phase space of 8, rather than 6, dimensions. One may worry that, in the enlarged picture, we should define another Hamiltonian to describe the evolution of the enlarged phase manifold. However, the dynamics of canonical coordinates that treat energy and time on an equal footing are typically described in the covariant formulation of Hamilton-Jacobi theory. Here, one considers the Hamilton function $S = S(x^\mu)$ with an affine evolution parameter τ. We will come back to the enlarged picture in a moment to describe its geometric consequences via the Kepler-Lorentz duality.

In using the moment map to construct functions on the phase manifold, let us reduce ourselves to the canonical coordinates x^i, y_i spanning 6-dimensions to treat the problem in Hamiltonian mechanics. We assert here without proof that the 6-dimensional submanifold of T^+M given by the nullity condition of the metric,

$$(g_\gamma)^{\mu\nu} y_\mu y_\nu = 0,$$

modulo H^1 or S^1, which we denote T_0^+M/H^1 or T_0^+M/S^1, is homeomorphic to T^+H^3 or T^+S^3, respectively [2]. Let us reduce our 8-dimensional cotangent bundle as such. The metric nullity condition alone yields the 7-dimensional submanifold T_0^+M and gives for the energy

$$y_0 = J_{(+)}(x^i, y_i) = \frac{1}{2} y(x^2 - 1)$$

for the repulsive scattering problem and

$$y_0 = -J_{(-)}(x^i, y_i) = -\frac{1}{2} y(x^2 + 1)$$

for the bound problem. The Hamiltonians J_+ and J_- defined in this section differs from that seen in the Moser method by a constant, but this should not matter since the constant nevertheless vanishes under the derivative in Hamilton's equations. If we wanted, we could have chosen the Hamiltonians to be the same as those in Chapter 3. From the Hamiltonians, calculating Hamilton's equations is straightforward

$$\frac{dx^i}{dx^0} = \mp \frac{\partial J_{(\pm)}}{\partial y_i}, \quad \frac{dy_i}{dx^0} = \pm \frac{\partial J_{(\pm)}}{\partial x^i}, \tag{5.12}$$

where we take the upper sign for repulsive scattering and the lower sign for the bound problem. Now, x^0 is just an evolution parameter corresponding to the Hamiltonians and parametrizes H^1 and S^1 in the respective cases. We can mod out H^1 or S^1 from T_0^+M by fixing any value for x^0. Fixing $x^0 = 0$, we get T_0^+M/H^1 and T_0^+M/S^1 homeomorphic to T^+H^3 and T^+S^3, respectively, which are cotangent bundles on which the Kepler problem are geodesic flows. We insert X^A, Y_A from the positive and negative energy results to get the functions in Eq. (5.9) of the moment map $\mu : T^+H^3 \to c^*(1,3)$ and

$\mu : T^+S^3 \to c^*(1,3)$

$$\begin{aligned} G_{-1,0} &= \pm\frac{1}{2}y(x^2+1) \\ G_{ij} &= y_i x_j - y_j x_i \\ G_{0,i} &= \pm y x_i \\ G_{-1,i} &= \frac{1}{2}(x^2+1)y_i - (x,y)x_i \\ G_{i,4} &= \frac{1}{2}(x^2-1)y_i - (x,y)x_i \\ G_{-1,4} &= (x,y) \\ G_{0,4} &= \pm\frac{1}{2}y(x^2-1). \end{aligned} \tag{5.13}$$

There is a canonical transformation taking (x^i, y_j, x^0, y_0) to the physical variables (q^i, p_j, t, p_t) of the Kepler problem. Though canonical transformation, it is possible to derive the Kepler Hamiltonian and show the equivalence of moment map functions in Eq. (5.13) and conserved physical quantities such as angular momentum and the LRL vector [2].

5.4.3 A Note on Geometry

The enlarged phase manifold $T^+M_{(+)} \simeq \frac{T^+(H^1\times H^3)}{\mathbb{Z}_2}$ of the scattering-state Kepler problem has an interesting geometry under stereographic projection. We have seen in Chapter 4 that both 4-momenta and 4-position of the scattering Kepler problem obey the invariance relation preserving H^3. Thus, the embedding spaces of H^3's for position and conjugate momenta have the same Minkowskian geometry $\mathbb{R}^{1,3}$. It follows that, whether we interpret it as the enlarged coordinate space or momentum space, the geometry of the manifold $M_{(+)} \simeq \frac{H^1\times H^3}{\mathbb{Z}_2}$ is the same. We treat the momentum space here. As we know from Fock and Moser, the stereographic projection of H^3 gives the Poincaré 3-ball $\mathcal{P}$, on which the physical momentum space curves of the scattering Kepler problem are geodesics starting and ending at the boundary. We also know that, topologically, H^1 is homeomorphic to $\mathbb{R}^1$. We thus have the homeomorphism $M_{(+)} \simeq \frac{\mathbb{R}^1\times\mathcal{P}}{\mathbb{Z}_2}$. That is, in the embedding momentum space $\mathbb{R}^{1,3}$ of the scattering-state Kepler problem, we are performing a stereographic projection of each of the H^3 submanifolds within the future null cone across varying energies and stacking them on top of each other. It is a curious fact that the result is a cylinder $\mathbb{R}^1 \times \mathcal{P}$ with conformal boundary, where the coordinates on the $\mathcal{P}$ are the physical momenta p_i and the coordinate along the length of the cylinder is the energy. In the momentum space of relativity related to the above by the Kepler-Lorentz duality, we obtain a similar hyperbolic cylinder performing a stereographic projection of each mass-shell and stacking them on top of each other. This is an alternative picture of the volume inside the future light cone.

As a final note, we discuss the Kepler-Lorentz duality in the light of recent connections between the symmetries of the Kepler problem and that of a maximally supersymmetric quantum field theory. In Ref. [22], the authors show that a relativistic quantum field theoretic extension of the dynamical (LRL) symmetry of the Kepler problem is the dual conformal symmetry in the momentum space of planar $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory. Though the authors only consider hydrogenic bound states of the Kepler problem with the symmetry group $SO(4)$, the sign of the coupling constant can be reversed to obtain canonical transformations corresponding to that of the scattering states with the symmetry group $SO(1,3)$. Indeed, we obtained in the previous section both the bound and scattering states of the Kepler problem by taking various cross-sections of the null cone in $\mathbb{R}^{2,4}$. In our construction in Chapter 4, we showed the equivalence of the momentum space of the scattering state Kepler problem and that of relativistic free particle in Minkowski spacetime, which are both symmetric under the Lorentz group $SO(1,3)$. If the high-energy analogue of the Kepler problem is the planar $\mathcal{N} = 4$ SYM with dual conformal symmetry, is there a such an analogue of the flat Minkowski spacetime?

The AdS/CFT correspondence (or gauge-gravity duality), originally formulated in the context of string theory, is the conjecture of a duality between quantum gravity in $(d+1)$-dimensions and quantum field theory at its d-dimensional conformal boundary in the large N limit of $SU(N)$ gauge theories [23–25]. The most well-studied case of this duality is between the 5-dimensional anti-de Sitter (AdS_5) space and the $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions (CFT_4). AdS_5 is defined as a generalized hyperboloid embedded in $\mathbb{R}^{2,4}$ and has the special property that, for a fixed-time spatial section, the ratio of its volume to the area of its boundary approaches a constant as one considers the entire section. Constrained at the conformal spatial boundary of (a cover of) AdS_5, the metric becomes that the 4-dimensional flat Minkowski spacetime of CFT_4.

There are a couple of qualitative features which suggest that a connection between the Kepler-Lorentz duality and gauge-gravity duality is not unreasonable: (1) They share the same symmetry groups. The isometry/conformal groups of $\mathrm{AdS}_5/\mathrm{CFT}_4$ and the (extended symmetry of) Kepler-Lorentz duality are both $SO(2,4)$. While the compactified Minkowski is the space of rays in the null cone of $\mathbb{R}^{2,4}$, AdS_5 is a hyperboloid in $\mathbb{R}^{2,4}$ which approaches the null cone as one takes the spatial coordinates to infinity. (2) In our construction of the Kepler-Lorentz duality, the scattering state Kepler problem represents a 1-dimensional reduction from relativistic kinematics via the isomorphism between geodesics (momentum space orbits) and their points of symmetry on the Poincaré 3-ball $\mathcal{P}$. That is, in the Kepler problem, we suppress its time dimension by considering the transformations of static orbits rather than of particles moving with respect to time. Hence, we get a $(d+1)$-dimensional spacetime kinematics from a d-dimensional scattering state Kepler problem.

If such a connection between the Kepler-Lorentz duality and gauge-gravity duality exists, it would be a remarkable example of a case where deep relationships in physics at the highest energy scales, where all particles travel at the speed of light, are realized even at the classical scales that are readily available to our senses. We hope that some of the accessible insights gained through the integrable structure of the two systems involved in the Kepler-Lorentz duality finds its use for gauge-gravity duality. This is a point of future investigation.

5.5 SU(2,2) and Twistors

It is well known that the Lorentz group $SO(1,3)$, which acts on 4-vectors, is doubly covered by the Möbius group $SL(2,\mathbb{C})$, which acts on spinors. One obtains spinors by expressing the 4-vector in a Hermitian form as a linear combination of Pauli matrices σ^μ and taking the "square root" to express the Hermitian matrix as a outer product of two spinors in $\mathbb{C}^2$. There is a conformal symmetry analogue of this relationship, as depicted in Fig. 1.1. The group $SO(2,4)$, which acts on "6-vectors" spanning four dimensions as seen in our retrieval of the Moser method, is doubly covered by the group $SU(2,2)$, which acts on twistors. **Twistors**, which can be understood as the higher dimensional analogue of spinors, belong in the space $\mathbb{C}^{2,2}$ and are composed of two spinors [2, 18].[11] It turns out, twistors and 6-vectors can be related by the Kustaanheimo-Stiefel map under stereographic coordinates. In this section, we will present the generators of the Lie algebra of $SU(2,2)$ for our choice of basis and introduce basic properties of twistors required to obtain the Kustaanheimo-Stiefel map. Instead of using the spinor index notation, we use the matrix notation in the mathematical physics literature following [2].

5.5.1 Generators of su(2,2)

In our choice of the basis in $su(2,2)$, the Lie algebra of $SU(2,2)$, it may seem natural to choose the quadratic form

$$\mathbf{H} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

due to the signature of the group $SU(2,2)$. Recall that the Lie algebra of (pseudo)unitary groups have (skew-)Hermitian elements, since the exponential of Hermitian matrices map to unitary matrices. Also, since the elements of special unitary groups have determinant 1, the elements of the corresponding algebra have trace 0 also via the exponential map.

[11] This property makes it sound like a Dirac spinor composed of two Weyl spinors, but spinors in a twistor are related in a nontrivial way unlike in a Dirac spinor.

For the generators $\mathbf{G}_{AB}$ of $su(2,2)$ under the form $\mathbf{H}$, the following relations are then satisfied:

$$\mathbf{G}_{AB}^{\dagger}\mathbf{H} + \mathbf{H}\mathbf{G}_{AB} = 0, \quad \mathrm{Tr}\ \mathbf{G}_{AB} = 0,$$

where the dagger $\dagger$ denotes the Hermitian adjoint. The first equation is the condition that generators with diagonal matrix elements are skew-Hermitian, while those with anti-diagonal elements are Hermitian. The generators $\mathbf{G}_{AB}$ also satisfy the Lie bracket relations

$$[\mathbf{G}_{AB}, \mathbf{G}_{AC}] = \eta_{AA}\mathbf{G}_{BC} \text{ or } = 0 \text{ if all indices different,}$$

similar to the generators $\mathcal{G}_{AB}$ of $so(2,4)$ in Eq. (5.2), and the two algebras are in fact isomorphic.

It is, however, more useful to consider the block-diagonal form $\mathbf{h}$ related to $\mathbf{H}$ by the transformation matrix

$$\mathbf{T} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix}$$

such that

$$\mathbf{H} \mapsto \mathbf{h} = (\mathbf{T}^{\dagger})^{-1}\mathbf{H}(\mathbf{T})^{-1} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{5.14}$$

and

$$\mathbf{G}_{AB} \mapsto \mathbf{g}_{AB} = \mathbf{T}\mathbf{G}_{AB}\mathbf{T}^{-1}.$$

The transformation takes us from the conformal extension of the Dirac basis to that of the Weyl basis of gamma matrices, as in the treatment of fermions in quantum field theory. The inner product $(\cdot,\cdot)$ is then defined

$$(\varphi, \psi) = \varphi^{\dagger}\mathbf{h}\psi,$$

where $\varphi, \psi \in \mathbb{C}^{2,2}$. It is also easy to prove from the relations above that

$$\mathbf{g}_{AB}^{\dagger}\mathbf{h} + \mathbf{h}\mathbf{g}_{AB} = 0, \quad \mathrm{Tr}\ \mathbf{g}_{AB} = 0, \tag{5.15}$$

as was the case for $\mathbf{G}_{AB}$. Let $\boldsymbol{\sigma}_i$ be the Pauli matrices

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which, together with the identity matrix, form the basis of 2×2 Hermitian matrices. We can now choose the 15 generators $\mathbf{g}_{AB}$ of $su(2,2)$ that satisfy Eq. (5.15) with the block-diagonal form $\mathbf{h}$:

$$\mathbf{g}_{jk} = \frac{1}{2}\begin{pmatrix} -\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k & 0 \\ 0 & -\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k \end{pmatrix}, \quad \mathbf{g}_{0j} = \frac{1}{2}\begin{pmatrix} -\boldsymbol{\sigma}_j & 0 \\ 0 & \boldsymbol{\sigma}_j \end{pmatrix},$$
$$\mathbf{g}_{-1,4} = \frac{1}{2}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix},$$
$$\mathbf{g}_{0,4} = \frac{1}{2}\begin{pmatrix} 0 & i\mathbb{1} \\ -i\mathbb{1} & 0 \end{pmatrix}, \quad \mathbf{g}_{j,4} = \frac{1}{2}\begin{pmatrix} 0 & i\boldsymbol{\sigma}_j \\ i\boldsymbol{\sigma}_j & 0 \end{pmatrix},$$
$$\mathbf{g}_{-1,0} = \frac{1}{2}\begin{pmatrix} 0 & i\mathbb{1} \\ i\mathbb{1} & 0 \end{pmatrix}, \quad \mathbf{g}_{-1,j} = \frac{1}{2}\begin{pmatrix} 0 & i\boldsymbol{\sigma}_j \\ -i\boldsymbol{\sigma}_j & 0 \end{pmatrix}.$$

They are conformal generators obtained via commutation relations of the conformal extension of the Weyl basis.

5.5.2 Twistors and Moment Map

We now introduce twistors [18]. An element $\psi \in \mathbb{C}^{2,2}$ such that

$$(\psi, \psi) = \psi^\dagger \mathbf{h} \psi = 0$$

are called **null twistors**. Let us denote the space of null twistors as T_0. We assume the equivalence relation $\psi \sim \psi e^{i\phi}$ with respect to a phase angle ϕ. Then, $T_0/\sim$, the space of null twistors modulo a phase transformation, is a 6 dimensional manifold. The manifold $T_0/\sim$ can be given the symplectic structure $\omega = d\theta$, where the canonical 1-form is

$$\theta = \mathrm{Im}(\psi, d\psi). \tag{5.16}$$

We will show in the next section that, given the relation between the two spinors in a twistor, that this is indeed the canonical 1-form providing the canonical transformation between the twistor and stereographic variables. Let us investigate how twistors transform under the action of $SU(2,2)$. In analogy to spinors, which transform like $\xi \mapsto \xi' = u\xi$ for $\xi \in \mathbb{C}^2$ and $g \in SL(2,\mathbb{C})$, twistors transform under the action of $SU(2,2)$ like

$$\psi \mapsto \psi' = g\psi, \tag{5.17}$$

for $g \in SU(2,2)$. Recall that spinors were obtained by (1) mapping a 4-vector to a 2×2 Hermitian matrix and (2) expressing the matrix as an outer product of two spinors in $\mathbb{C}^2$. If the determinant of the Hermitian matrix is 0, one can express the matrix of rank 1 as an outer product of two identical spinors. Similarly, the outer products of twistors form an object of relevance. Let us define such an object

$$\boldsymbol{\mu}(\psi) = -i\psi\psi^\dagger \mathbf{h}, \quad \text{with } \psi^\dagger \mathbf{h}\psi = 0, \tag{5.18}$$

where presence of terms aside from the outer product $\psi\psi^\dagger$ will be clarified in a moment. Note the following properties of $\boldsymbol{\mu}(\psi)$:

i $\boldsymbol{\mu}(\psi)$ satisfies

$$\boldsymbol{\mu}^\dagger(\psi)\mathbf{h} + \mathbf{h}\boldsymbol{\mu}(\psi) = 0, \quad \mathrm{Tr}\ \boldsymbol{\mu}(\psi) = -i\psi^\dagger \mathbf{h}\psi = 1,$$

as do the generators $\mathbf{g}_{AB}$ in Eq. (5.15), so we have $\boldsymbol{\mu}(\psi) \in su(2,2)$. But it happens that the generators $\mathbf{g}^*_{AB}$ of the dual algebra $su^*(2,2)$ can be found from $\mathbf{g}_{AB}$ by simply changing the sign of the generators of noncompact transformations. Hence, $su(2,2) = su^*(2,2)$ and it follows that $\boldsymbol{\mu}(\psi) \in su^*(2,2)$. Therefore, $\boldsymbol{\mu}(\psi)$ is the moment map $\boldsymbol{\mu} : T_0/\sim \to su^*(2,2)$! Since we took $T_0/\sim$ as the domain, the choice of a phase is modded out and the moment map $\boldsymbol{\mu}(\psi)$ is thus one-to-one.

ii $\boldsymbol{\mu}(\psi)$ transforms under $SU(2,2)$ like the Hermitian form of the 4-vector transforms under $SL(2,\mathbb{C})$. If $g \in SU(2,2)$, we have $g^\dagger \mathbf{h} g = \mathbf{h}$ which preserves the metric by definition. Then, it follows that $g^{-1} = \mathbf{h}^{-1} g^\dagger \mathbf{h}$ and so $\boldsymbol{\mu}(\psi)$ transforms as

$$\boldsymbol{\mu} \mapsto \boldsymbol{\mu}' = g\boldsymbol{\mu}g^{-1} = -ig\psi\psi^\dagger \mathbf{h}\mathbf{h}^{-1} g^\dagger \mathbf{h} = -i(g\psi)(g\psi)^\dagger \mathbf{h}.$$

That is, the action of $SU(2,2)$ splits into simpler actions on each null twistor ψ while preserving the general form of the original expression of $\boldsymbol{\mu}(\psi)$. This is analogous to the case of spinors, where the action of $SL(2,\mathbb{C})$ splits into simpler actions on each null spinor ψ.

The moment map $\boldsymbol{\mu}(\psi)$ can be decomposed in terms of the generators $\mathbf{g}^*_{AB}$. The coefficients of the generators will give us the smooth functions on $T_0/\sim$. We will make use of this property in the following section.

5.6 Connection to Kustaanheimo-Stiefel Method

It turns out, null twistors in $\mathbb{C}^{2,2}$ and 6-vectors in $\mathbb{R}^{2,4}$ can be related by the Kustaanheimo-Stiefel map [2, 14]. Recall from Sec. 5.3 and 5.4 that the dimensions of manifold $\mathcal{S}$ of null simple bivectors is only 8, not 12, so the cotangent bundle T^+M_γ of cross sections at various energies were expressible in terms of the canonical coordinates x^μ and y_μ in place of X^A and Y_A. This was further reduced by choosing the nullity condition $(g_\gamma)^{\mu\nu} y_\mu y_\nu = 0$, which is preserved under the conformal action. This allowed us to express y_0, interpreted now as the Hamiltonian, in terms of x^i and y_i and to fix an arbitrary value for the evolution parameter x^0. We have left x^i and y_i, which are in fact canonical coordinates on the phase space of the Kepler problem $T^+\mathbb{R}^3$. In this section, we present the Kustaanheimo-Stiefel map and its inverse relating the space of null twistors $T_0/\sim$ to the phase space of the Kepler problem $T^+\mathbb{R}^3$. The Kustaanheimo-Stiefel map shows the equivalence of the 3d Kepler problem and the 4d harmonic oscillator, for both bound and scattering states.

5.6.1 Inverse Map

Let us define two Hermitian matrices

$$\mathbf{x} = x^i \boldsymbol{\sigma}_i = \begin{pmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{pmatrix} \tag{5.19}$$

$$\mathbf{y} = y\mathbb{1} + y_i \boldsymbol{\sigma}^i = \begin{pmatrix} y + y_3 & y_1 - iy_2 \\ y_1 + iy_2 & y - y_3 \end{pmatrix}, \tag{5.20}$$

where $y = \sqrt{y^i y_i}$ and x^i and y_i are canonical coordinates on $T^+\mathbb{R}^3$. Since the determinant of $\mathbf{y}$ is 0, we can express $\mathbf{y}$ as the outer product of two identical spinors $\xi = \xi(y_i)$:

$$\mathbf{y} = \xi(y_i)\xi^\dagger(y_i).$$

Let us take a generic null twistor

$$\psi = \begin{pmatrix} z \\ iw \end{pmatrix}, \quad z \neq 0 \tag{5.21}$$

where $z, w \in \mathbb{C}^2$ are each spinors. In what is called the **incidence relation** in twistor theory [18], z and w are related by a Hermitian matrix. If we take $z = \xi(y_i)$ and $w = \mathbf{x}\xi(y_i)$, we have the map $\psi(x^i, y_i) : T^+\mathbb{R}^3 \to T_0/\sim$, given by

$$\psi(x^i, y_i) = \begin{pmatrix} \xi(y_i) \\ i\mathbf{x}\xi(y_i) \end{pmatrix}. \tag{5.22}$$

This is the inverse Kustaanheimo-Stiefel map. One can verify this by checking that one gets the identity when substituted into the forward map, which we will obtain shortly.

Before proceeding, let us check that the canonical 1-forms of $T_0/\sim$ and $T^+\mathbb{R}^3$ are equal to show that the map is indeed a canonical transformation or, equivalently in the mathematical physics literature, a **symplectomorphism**. That is, now that we know about the pullback map, we check that the pullback ψ^* of the canonical 1-form θ in Eq.

(5.16) gives us the 1-form $y_i dx^i$:

$$\begin{aligned}\psi^*\theta &= \mathrm{Im}\left[\begin{pmatrix}\xi^\dagger & -i\xi^\dagger\mathbf{x}\end{pmatrix}\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\begin{pmatrix}d\xi\\ id\mathbf{x}\xi + i\mathbf{x}d\xi\end{pmatrix}\right]\\ &= \xi^\dagger d\mathbf{x}\xi \;=\; \mathrm{Tr}\mathbf{y}d\mathbf{x} \;=\; y_i dx^i. \qquad \checkmark\end{aligned}$$

It it worthwhile to mention that the use of stereographic coordinates in the Hermitian matrix $\mathbf{x}$ of the incidence relation, as in Eq. (5.19), is unconventional in the context of twistor theory. Instead of using stereographic coordinates in $\mathbb{R}^3$, it is typical to let $\mathbf{x}$ be the Hermitian representation of a spacetime 4-vector in the flat Minkowski space $\mathbb{R}^{1,3}$. If one can somehow limit the 4-vectors to the invariant submanifold H^3 of $\mathbb{R}^{1,3}$, it is possible to take stereographic coordinates in $\mathbb{R}^3$ to get the Kustaanheimo-Stiefel map as we have done above. In the spirit of the Moser method, this is done here by obtaining $\mathbf{x}$ in the momentum space, where $x^1, x^2, x^3 \in \mathbb{R}^3$ are coordinates obtained via a stereographic projection of the invariant mass shell H^3. We will in fact perform a canonical exchange between x^i and y_i in the following section.

5.6.2 Kustaanheimo-Stiefel Map

Let us take a generic form of a null twistor in Eq. (5.21). Then, the moment map $\boldsymbol{\mu}(\psi)$ in Eq. (5.18) becomes

$$\boldsymbol{\mu}(z,w) = \begin{pmatrix} zw^\dagger & izz^\dagger\\ iww^\dagger & -wz^\dagger\end{pmatrix}. \tag{5.23}$$

Recall that moment maps act on generators of a symmetry and yields smooth functions on the phase manifold, given that the phase manifold is preserved under such symmetry. Let us take the moment map $\boldsymbol{\mu} : T_0/\sim\to su^*(2,2)$ given by

$$\boldsymbol{\mu}(z,w) = \sum_{A<B} g_{AB}(z,w)\mathbf{g}^*_{AB} \tag{5.24}$$

where $g_{AB}(z,w)$ are functions on $T_0/\sim$ and the moment map $\boldsymbol{\nu} : T^+\mathbb{R}^3 \to su^*(2,2)$ given by

$$\boldsymbol{\nu}(x^i, y_i) = \sum_{A<B} G_{AB}(x^i,y_i)\mathbf{g}^*_{AB} \tag{5.25}$$

where $G_{AB}(x^i, y_i)$ are functions on $T^+\mathbb{R}^3$ obtained in Eq. (5.13). Comparing the functions $g_{AB}(z,w)$ and $G_{AB}(x^i,y_i)$, we can retrieve the Kustaanheimo-Stiefel map $\mathbb{K} : T_0/\sim\to T^+\mathbb{R}^3$ seen in Chapter 3.

To find the functions $g_{AB}(z,w)$, let us project the moment map $\boldsymbol{\mu}(\psi) = -i\psi\psi^\dagger \mathbf{h}$ on to the basis $\mathbf{g}^*_{AB}$ of Eq. (5.24). If we multiply the relation

$$\sum_{A<B} g_{AB}(z,w)\mathbf{g}^*_{AB} = -i\psi\psi^\dagger \mathbf{h}$$

to the right by $\mathbf{g}_{CD}$ and take the trace, we get

$$g_{AB}(z,w) \;=\; i\mathrm{Tr}\; \psi\psi^\dagger \mathbf{h}\mathbf{g}_{AB} \;=\; i\psi^\dagger \mathbf{h}\mathbf{g}_{AB}\psi.$$

Explicitly,

$$\begin{aligned}
g_{-1,0} &= -\frac{1}{2}(z^\dagger z + w^\dagger w) \\
g_{ij} &= \mathrm{Re}(z^\dagger \boldsymbol{\sigma}_i \boldsymbol{\sigma}_j w) \\
g_{0i} &= -\mathrm{Re}(z^\dagger \boldsymbol{\sigma}_i w) \\
g_{-1,i} &= \frac{1}{2}(z^\dagger \boldsymbol{\sigma}_i z - w^\dagger \boldsymbol{\sigma}_i w) \\
g_{i4} &= -\frac{1}{2}(z^\dagger \boldsymbol{\sigma}_i z + w^\dagger \boldsymbol{\sigma}_i w) \\
g_{-1,4} &= \mathrm{Re}(z^\dagger w) \\
g_{0,4} &= \frac{1}{2}(z^\dagger z - w^\dagger w).
\end{aligned} \tag{5.26}$$

Comparing $g_{AB}(z,w)$ with $G_{AB}(x^i, y_i)$, taking the upper sign for the scattering problem and the lower sign for the bound problem, we have

$$-G_{i4} + G_{-1,i} = y_i, \qquad -g_{k4} + g_{-1,i} = z^\dagger \boldsymbol{\sigma}_i z,$$

thus

$$y_i = z^\dagger \boldsymbol{\sigma}_i z, \qquad y = z^\dagger z. \tag{5.27}$$

Comparing also

$$G_{0i} = \pm y x^i \qquad g_{0k} = -\mathrm{Re}(z^\dagger \boldsymbol{\sigma}_i w),$$

we have

$$x_i = \frac{\mp\mathrm{Re}(z^\dagger \boldsymbol{\sigma}_i w)}{z^\dagger z}. \tag{5.28}$$

If we insert

$$z = \begin{pmatrix} z_1 + iz_2 \\ z_3 + iz_4 \end{pmatrix}, \quad w = \begin{pmatrix} w_1 + iw_2 \\ w_3 + iw_4 \end{pmatrix}, \quad z_1, \cdots, w_4 \in \mathbb{R}$$

and perform a canonical exchange between coordinates and momenta

$$x^i \to -y_i, \quad y_j \to x^j,$$

Eqs. (5.26) and (5.27) give the Kustaanheimo-Stiefel map $\mathbb{R} : T_0/\sim \to T^+\mathbb{R}^3$ for the scattering and bound problems, respectively. The constraint of the domain variables in Eq. (3.23) comes simply from the null twistor condition, which can be worked out directly from the inverse map $\mathbb{K}^{-1}$. The 1-dimensional kernel follows from the equivalence relation $\psi \sim \psi e^{i\phi}$. Since the Kustaanheimo-Stiefel map in the attractive case is the canonical extension of the Hopf map $S^3 \xrightarrow{S^1} S^2$, the kernel is indeed the equivalence relation of the elements of the S^1 fiber attached to S^2. One obtains the Hamiltonian for the respective problems by comparing $G_{0,4}$ with $g_{0,4}$ and $G_{-1,0}$ with $g_{-1,0}$, taking appropriate signs. Recalling that the regularized Hamiltonians take the form $J_{(+)} = \frac{1}{2}y(x^2-1)$ in the scattering case and that $J_{(-)} = \frac{1}{2}y(x^2+1)$ in the bound case, we find

$$\begin{aligned} J_{(+)}(z,w) &= \frac{1}{2}(|z|^2 - |w|^2), \\ J_{(-)}(z,w) &= \frac{1}{2}(|z|^2 + |w|^2). \end{aligned} \tag{5.29}$$

which are indeed the Hamiltonians for the inverted and simple 4D harmonic oscillators. Hence, the bound and scattering 3D Kepler problems are equivalent to the simple and inverted 4D harmonic oscillators via the Kustaanheimo-Stiefel map $\mathbb{K}$.

The results of this section, together with the Kepler-Lorentz duality, suggest the equivalence of the 3D scattering state Kepler problem, 4D inverted harmonic oscillator, and the relativistic free particle. Although the Kepler-Lorentz duality is, in hindsight, more apparent since the two systems share the same symmetry group $SO(1,3)$, the equivalence of the 4D inverted harmonic oscillator and relativistic free particle is more subtle. Along the lines of observations made at the end of Sec. 5.4, studies of 2-dimensional string theory show that a toy version of AdS/CFT correspondence allows one to recover all of the physics of 2D string theory by considering an equivalent system of non-interacting fermions in an inverted harmonic potential [26, 27]. Also in the recent years, the study of scattering amplitudes in the twistor framework has yielded many fruitful insights into the structure of scattering amplitudes in gauge theories and gravity [28–34]. As Kepler-Lorentz duality can be seen as a classical analogue of $\mathrm{AdS}_5/\mathrm{CFT}_4$ sharing the same isometry/symmetry groups, it remains to be seen whether there exists some analogous version of the 3-equivalence involving twistors at high energies. The construction of a dictionary among the quantities in the three systems in a tractable classical framework may provide insights into such systems.

5.7 Concluding Remarks

In this chapter, we highlighted a few possible directions in which the Kepler-Lorentz duality can be extended, along with some basic background required for such an investigation. We saw that by extending the symmetry of the Kepler problem from dynamical symmetry to conformal symmetry, we were able to treat the positive and negative energy problems in a unified picture as cross-sections of a greater manifold. The symmetries of the enlarged phase manifold, along with insights from recent studies of momentum space symmetries of a toy quantum field theory, allowed us to suggest a possible connection between the Kepler-Lorentz duality and gauge-gravity duality. We also found that a cover group of the conformal group, which acts on twistors, provides a natural way to show that 3D scattering state Kepler problem is equivalent to a 4d inverted harmonic oscillator.

It is surprising that the Kepler-Lorentz duality contains some qualitative features that seem to reflect those seen in more fundamental frameworks such as string theory and twistor theory. We hope that some of the tractable, simple insights gained through the Kepler-Lorentz duality may find its use in thinking about such theories.

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