Moduli of local systems and flat connections on smooth varieties

Tony Pantev

University of Pennsylvania

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Outline

- joint with Bertrand Töen
- Study the geometry of the moduli of:
  - flat connections, or
  - local systems

  on a smooth non-proper X/k, $\operatorname{char} k = 0$, with a view towards
  - Constructing (shifted) Poisson structures, and
  - Describing their symplectic leaves.

Motivation

X - compact oriented topological surface,
G - a complex reductive group.

Classical story: Fock-Rosly, Goldman, Guruprasad-Rajan, Guruprasad-Huebschmann-Jeffrey-Weinstein, ...

- The moduli $M_G(X)$ of $\rho : \pi_1(X, x) \to G$ has an algebraic Poisson structure;
- The symplectic leaves in $M_G(X)$ are moduli spaces of ρ with fixed monodromy at infinity.

Goal: Extend these statements to higher dimensional smooth varieties X.

Main results (i)

Fix a field k of char$k = 0$

Theorem: [P-Töen] Let X be a d-dimensional smooth complex algebraic variety and let G be a reductive algebraic group over k. Then

(1) The derived moduli stack $Loc_G(X)$ of G-local systems on X has a natural $(2-2d)$-shifted Poisson structure.

(2) This shifted Poisson structure admits generalized symplectic leaves. Among those are the derived moduli of G local systems with fixed monodromy at infinity.

Main results (ii)

Comments:

- When $d = 1$ the Poisson structure in **(1)** specializes to Goldman's Poisson structure on the moduli of representations $\pi_1(X, x) \to G$.
- **(2)** is tricky: need to understand how to fix local monodromies in the derived setting. Subtle issues:
  - can not be seen on $t_0 Loc_G(X)$ and involves higher homotopy coherences;
  - an additional constraint - **strictness** - has to be imposed on the local monodromies at infinity.

Main results (iii)

Theorem: [P-Töen] Let X be a d-dimensional smooth algebraic variety over k. Then

(1) The derived moduli stack $\mathrm{Vect}^{\nabla}(X)$ of flat vector bundles on X has a natural $(2-2d)$-shifted Poisson structure.

(2) There is a well defined derived stack of flat bundles $\mathrm{Vect}^{\nabla}(\widehat{\partial}X)$ on the formal boundary of X. The shifted Poisson structure of **(1)** is realized as a Lagrangian structure on the restriction map $R : \mathrm{Vect}^{\nabla}(X) \to \mathrm{Vect}^{\nabla}(\widehat{\partial}X)$.

(3) The fiber of R over a flat vector bundle on $\widehat{\partial}X$ is a derived algebraic space locally of finite presentation.

Main results (iv)

Comments:

- The formal boundary $\widehat{\partial} X$ should encode the punctured formal neighborhood of the boundary divisor in a good compactification $X \subset \mathfrak{X}$.
- Rigid analytic and non-commutative models for $\widehat{\partial} X$ have been considered in **[Ben-Bassat-Temkin], [Efimov], [Hennion-Porta-Vezzosi]**. Upshot: $\widehat{\partial} X$ has a well defined sheaf theory and a well defined stack $\mathrm{Perf}(\widehat{\partial} X)$ of perfect complexes.

Main results (v)

Comments:

- The bulk of the work goes into constructing a derived stack $\text{Perf}^{\nabla}(\widehat{\partial}X)$ of perfect complexes equipped with flat connections on $\widehat{\partial}X$ (studied in depth in **[Raskin]** for $X = \mathbb{A}^1$).
- The stacks $\text{Vect}^{\nabla}(X)$ and $\text{Vect}^{\nabla}(\widehat{\partial}(X))$ are not algebraic but are formally representable at field valued points. This is crucial for defining symplectic, Lagrangian, and Poisson structures.
- The existence of the Lagrangian structure on $R : \text{Vect}^{\nabla}(X) \to \text{Vect}^{\nabla}(\widehat{\partial}X)$ boils down to Poincaré duality for compactly supported cohomology relative to various derived base schemes.

Moduli of local systems (i)

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G-local systems on X

locally constant principal G-bundles on X

Moduli of local systems (i)

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Main object of study: The moduli stack $Loc_G(X)$ of G-local systems on X

Moduli of local systems (ii)

Properties:

- $Loc_G(X)$ is a derived Artin stack over k.
- $t_0 Loc_G(X)$ depends only on the fundamental group of X. It is the moduli stack of representations of $\pi_1(X, x)$ into G, i.e.

$$t_0 Loc_G(X) = \mathcal{M}_G(X) = [R_G(\pi_1(X, x))/ G]$$

Here $R_G(\pi_1(X, x))$ is the **character scheme** of X: the affine k-scheme representing the functor

$$R_G(\pi_1(X, x)) : \quad \text{commalg}_k \longrightarrow \text{Sets},$$
$$A \longrightarrow \text{Hom}_{\text{grp}}(\pi_1(X, x), G(A)).$$

Moduli of local systems (iii)

Properties:

- The stack $\mathcal{M}_G(X) = t_0 Loc_G(X)$ has a course moduli space which is the affine GIT quotient

$$M_G(X) = R_G(X)/\!\!/G,$$

and

$$M_G(X)(k) = \left(\begin{array}{l} \text{conjugacy classes of } \rho : \pi_1(X,x) \to G \\ \text{with } \overline{\mathrm{im}(\rho)}\text{-reductive} \end{array} \right)$$

$$= \left(\begin{array}{l} \text{iso classes of locally constant } G(k) \\ \text{bundles on } X \end{array} \right)$$

- In general the derived structure on $Loc_G(X)$ **depends** on the full homotopy type of X.

Shifted symplectic structures

Recall: [PTVV]

- If F is derived Artin locally f.p. over k we have a **complex of closed 2-forms** $\mathcal{A}^{2,cl}(F)$ on F.
- When $F = \mathrm{RSpec} A$, then $\mathcal{A}^{2,cl}(F)$ corresponds to the module $\mathrm{tot}^{\prod}(F^p(A)[p])$.
- An n-cocycle ω in the complex $\mathcal{A}^{2,cl}(F)$ is a **closed n-shifted 2-form**.
- ω is an ***n*-shifted symplectic structure** if the contraction $\omega^\flat : \mathbb{T}_F \xrightarrow{\sim} \mathbb{L}_F$ with the induced element in $H^n(F, \wedge^2\mathbb{L}) = H^n(\mathcal{A}^{2,cl}(F))$ is a quasi-iso.

Relative structures

Let $f : F \to F'$ be a morphism between derived Artin stacks over k, then

- An $(n-1)$-shifted **isotropic structure** on f is a pair (ω, h), where ω is an n-shifted symplectic structure on F', and h is a homotopy between $f^*(\omega)$ and 0 inside the complex $\mathcal{A}^{2,cl}(F)$.
- An isotropic structure (ω, h) is **Lagrangian** if moreover the canonical induced morphism $h^\flat : \mathbb{T}_f \xrightarrow{\sim} \mathbb{L}_F[n-1]$ is a quasi-isomorphism.
  Note: An $(n-1)$-shifted Lagrangian structure on $f : F \to \operatorname{Spec} k$ is simply an $(n-1)$-shifted symplectic structure on F.

Structures on $Loc_G(X)$ (i)

$(X, \partial X)$ - compact oriented topological manifold of dim $= d$
G - a reductive algebraic group over k.

Theorem:

(a) [PTVV] If $\partial X = \varnothing$, then the derived stack $Loc_G(X)$ has a $(2-d)$-shifted symplectic structure which is canonical up to a choice of a non-degenerate element in $(\mathrm{Sym}^2 \mathfrak{g}^\vee)^G$

(b) [Calaque] The restriction map $Loc_G(X) \longrightarrow Loc_G(\partial X)$ carries a canonical $(2-d)$-shifted Lagrangian structure for the $3-d = 2-(d-1)$-shifted symplectic structure on the target.

Structures on $Loc_G(X)$ (ii)

Note: When X is a Riemann surface with boundary we recover the symplectic structures on moduli of G-local systems on X with prescribed monodromies at infinity (usually constructed by quasi-Hamiltonian reduction).

Structures on $Loc_G(X)$ (ii)

Example: Suppose $(X, \partial X)$ is an oriented surface with boundary. Then

- ∂X is a disjoint union of oriented circles, and so $Loc_G(\partial X) \simeq \prod [G/G]$ where $[G/G]$ denotes the stack quotient of the conjugation action of G on itself.
- The stack $Loc_G(S^1) = [G/G]$ carries a canonical 1-shifted symplectic structure.
- For any $\lambda \in G$, the inclusion of the conjugacy class $\mathbb{O}_\lambda \subset G$ of λ gives a canonical Lagrangian structure on the map $BG_\lambda \simeq [\mathbb{O}_\lambda/G] \hookrightarrow [G/G]$.

Structures on $Loc_G(X)$ (iii)

Assigning elements $\lambda_i \in G$ to each boundary component of X, we get two 0-shifted Lagrangian morphisms

$$\begin{array}{ccc} \prod BG_{\lambda_i} & & Loc_G(X). \\ & \searrow \quad \swarrow & \\ & \prod [G/G] & \end{array}$$

By **[PTVV]** the fiber product of these two maps has a canonical 0-shifted symplectic structure. This fiber product, is the derived stack

$$Loc_G(X, \{\lambda_i\})$$

of G-local systems on X whose local monodromies at infinity are belong to the conjugacy classes $\{\mathbb{O}_{\lambda_i}\}$.

Shifted Poisson structures (i)

Recall: [CPTVV]

- For F a derived Artin stack$/k$, can form the dg Lie algebra of ***n*-shifted polyvector fields** $\Gamma(F, \mathrm{Sym}_{\mathcal{O}}(\mathbb{T}_F[-n-1]))[n+1]$.
- An ***n*-shifted Poisson structure** on F is a morphism in the ∞-category of graded dg-Lie algebras

$$p : k[-1](2) \longrightarrow \Gamma(F, \mathrm{Sym}_{\mathcal{O}}(\mathbb{T}_F[-n-1]))[n+1],$$

where $k[-1](2)$ is the graded dg Lie algebra which is k placed in homological degree 1 and grading degree 2, equipped with the zero Lie bracket.

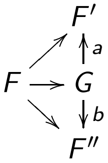
Shifted Poisson structures (ii)

Remark: **[Melani-Safronov,Costello-Rozenblyum,Nuiten]**
Shifted Poisson structures can always be described in terms of shifted symplectic groupoids (Weinstein program).

Shifted Poisson structures (ii)

Theorem: [Costello-Rozenblyum] If F is a derived Artin stack the space of n-shifted Poisson structure on F is weakly equivalent to the space of equivalence classes of n-shifted Lagrangian maps $F \to F'$ to formal derived stacks F'.

Note: $[F \to F'] \sim [F \to F'']$ if there exists an n-shifted Lagrangian map $F \to G$ and a commutative diagram



with a and b formally étale and compatible with the Lagrangian structures.

Shifted Poisson structures (iii)

Example: For a compact oriented d-dimensional manifold X with boundary ∂X, the restriction map

$$Loc_G(X) \longrightarrow Loc_G(\partial X)$$

is Lagrangian **[Calaque]** and so can be viewed as a $(2-d)$-shifted Poisson structure on $Loc_G(X)$.

Simplectic leaves (i)

Classically a Poisson structure on a smooth variety induces a foliation of the variety by symplectic leaves.
For an n-shifted Poisson structure on a derived stack F given by a Lagrangian map $f : F \to F'$, the symplectic leaves are the appropriately interpreted fibers of f.

Definition: A **generalized symplectic leaf** of F is a derived stack of the form $F \times_{F'} \Lambda$ for any n-shifted Lagrangian morphism $\Lambda \to F'$

Note: By **[PTVV]** a generalized symplectic leaf carries a canonical n-shifted symplectic structure.

Simplectic leaves (ii)

Example: X - a compact oriented surface with boundary. The restriction map

$$Loc_G(X) \longrightarrow Loc_G(\partial X) = \prod [G/G]$$

carries a 0-shifted Lagrangian structure and thus corresponds to a 0-shifted Poisson structure on $Loc_G(X)$.

$Loc_G(X, \{\lambda_i\})$ - the derived moduli stack of G-local systems on X with fixed monodromies at infinity - is a generalized symplectic leaf in $Loc_G(X)$.

Betti spaces - theorems (i)

The **boundary of a topological space** Y is the pro-homotopy type $\partial Y := \lim_{K \subset Y} (Y - K) \in \mathrm{Pro}(\mathbb{T})$.

Betti spaces - theorems (i)

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taken in the ∞-category $\mathbb{T}$ of homotopy types and over the opposite category of compact subsets $K \subset Y$

Betti spaces - theorems (i)

The **boundary of a topological space** Y is the pro-homotopy type $\partial Y := \lim_{K \subset Y} (Y - K) \in \mathrm{Pro}(\mathbb{T})$.

Note: The pro-object ∂Y is in general not constant and can be extremely complicated. However if $X = Z(\mathbb{C})$ for a smooth n-dimensional complex algebraic variety Z, we have:

> **Proposition:** The pro-object ∂X is equivalent to a constant pro-object in $\mathbb{T}$ which has the homotopy type of a compact oriented topological manifold of dimension $2n - 1$.

Remark: ∂X has the homotopy type of the biundary of the simple real oriented blowup of a good compactification of Z along its normal crossing boundary.

Betti spaces - theorems (ii)

Suppose $X = Z(\mathbb{C})$ for a smooth n-dimensional complex algebraic variety Z, then

Claim: The canonical map $\partial X \longrightarrow X$ induces a restriction morphism of derived locally f.p. Artin stacks

$$r : Loc_G(X) \longrightarrow Loc_G(\partial X).$$

which is equipped with a canonical $(2-2n)$-shifted Lagrangian structure with respect to the canonical shifted symplectic structure on $Loc_G(\partial X)$.
In particular r can be viewed as a $(2-2n)$-shifted Poisson structure on $Loc_G(X)$.

Symplectic leaves - smooth D (i)

Assume Z admits a smooth compactification $Z \subset \mathfrak{Z}$ with $D = \mathfrak{Z} - Z = \coprod_i D_i$ a smooth divisor. Then

- $\partial X =\sim$ (oriented circle bundle over D) classified by elements $\alpha_i \subset H^2(D_i, \mathbb{Z})$, $\alpha_i = c_1(N_{D_i/\mathfrak{Z}})$.
- Given $\lambda_i \in G$ with centralizer Z_i, the group S^1 acts on BZ_i (via λ_i) and naturally on $[G/G]$ so that the Lagrangian structure on the map $BZ_i \to [G/G]$ is S^1-equivariant.
- Twisting by α_i gives a 1-shifted Lagrangian morphism

$$(\dagger_i) \qquad {}_{\alpha_i}\widetilde{BZ}_i \longrightarrow {}_{\alpha_i}\widetilde{[G/G]}$$

of locally constant families of derived Artin stacks over D_i.

Symplectic leaves - smooth D (ii)

Passing to global sections gives moduli stacks:

$$Loc_G(\partial_i X) = \mathrm{Map}\left(\partial_i X, BG\right) = \Gamma\left(D_i, {}_{\alpha_i}\widetilde{[G/G]}\right);$$

$$Loc_{Z_i,\alpha_i}(D_i) = \Gamma\left(D_i, {}_{\alpha_i}\widetilde{BZ_i}\right)$$

Symplectic leaves - smooth D (ii)

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G local systems on the component $\partial_i X$ of ∂X mapping tp D_i

Symplectic leaves - smooth D (ii)

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Z_i local systems on D_i twisted by α_i

Symplectic leaves - smooth D (ii)

Passing to global sections gives moduli stacks:

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$$Loc_{Z_i,\alpha_i}(D_i) = \Gamma\left(D_i, {}_{\alpha_i}\widetilde{BZ_i}\right)$$

Since D_i is a compact topological manifold endowed with a canonical orientation the map $(\dagger_i)$ induces a $(3-2n)$-shifted Lagrangian morphism of derived Artin stacks

$$r_i : Loc_{Z_i,\alpha_i}(D_i) \longrightarrow Loc_G(\partial_i X).$$

Symplectic leaves - smooth D (iii)

Combining all r_i we get a $(3-2n)$-shifted Lagrangian morphism

$$r = \prod_i r_i : \prod_i Loc_{Z_i,\alpha_i}(D_i) \longrightarrow \prod_i Loc_G(\partial_i X) = Loc_G(\partial X).$$

By the Lagrangian intersection theorem **[PTVV]** the fiber product of derived stacks

$$Loc_G(X, \{\lambda_i\}) := \left(\prod_i Loc_{Z_i,\alpha_i}(D_i) \right) \underset{Loc_G(\partial X)}{\times} Loc_G(X)$$

has a canonical $(2-2n)$-shifted symplectic structure.

Symplectic leaves - smooth D (iv)

By construction

- $Loc_G(X, \{\lambda_i\})$ is the derived stack of G-local systems on X whose local monodromy around D_i is fixed to be in the conjugacy class $\mathbb{O}_{\lambda_i}$ of λ_i.
- The natural map

$$Loc_G(X, \{\lambda_i\}) \longrightarrow Loc_G(X)$$

realizes $Loc_G(X, \{\lambda_i\})$ as a generalized symplectic leaf of the $(2-2n)$-shifted Poisson structure on $Loc_G(X)$.

This proves part **(2)** of the Main theorem in the Betti setting.

Symplectic leaves - two components (i)

Assume $D = \mathfrak{Z} - Z = D_1 \cup D_2$ has two smooth irreducible components meeting transversally at a smooth D_{12}. Then

$$\partial X \simeq \partial_1 X \coprod_{\partial_{12} X} \partial_2 X.$$

where $\partial_i X$ is an oriented circle bundle over $D_i^o = D_i - D_{12}$, and $\partial_{12} X$ is an oriented $S^1 \times S^1$-bundle over D_{12}.

Note: Each $\partial_i X$ has the homotopy type of an oriented compact manifold of dimension $2n - 1$ with boundary canonically equivalent to $\partial_{12} X$.

Symplectic leaves - two components (ii)

Theorem: [P-Töen]

(i) For a commuting pair of elements $(\lambda_1, \lambda_2) \in G \times G$ the map

$$Loc_G(\partial_1 X, \lambda_1) \underset{Loc_G(\partial_{12} X)}{\times} Loc_G(\partial_2 X, \lambda_2) \longrightarrow Loc_G(\partial X) \times Loc_G(\partial_{12} X, \{\lambda_1, \lambda_2\})$$

comes equipped with a natural Lagrangian structure.

(ii) If moreover the pair (λ_1, λ_2) is strict then the derived Artin stack

$$Loc_G(X, \{\lambda_1, \lambda_2\})$$

comes equipped with a natural $(2-2n)$-shifted symplectic structure which is a symplectic leaf of $Loc_G(X)$.

Perfect complexes with flat connections (i)

Suppose X is a smooth variety over k, and let X_{DR} be the de Rham functor of X, i.e. the (discrete, underived) stack

$$X_{DR}: \quad \mathrm{cdga}_k^{\leqslant 0} \longrightarrow \mathrm{Sets} \subset \mathrm{SSets}$$
$$A \longrightarrow X\left(\mathrm{Spec}\left(A_{\mathrm{red}}\right)\right)$$

The **derived stack of perfect complexes with flat connections** on X is by definition

$$\boxed{\mathrm{Perf}^{\nabla}(X) = \mathrm{Map}_{\mathrm{dSt}_k}(X_{DR}, \mathrm{Perf})}$$

Perfect complexes with flat connections (ii)

If X is not proper $\mathrm{Perf}^{\nabla}(X)$ is not representable. However, since X is a finite colimit of affine k-schemes and $\mathrm{Perf}^{\nabla}(X)$ is a mapping stack one checks that the stack $\mathrm{Perf}^{\nabla}(X)$ has good infinitesimal properties:

Proposition: Let X be a smooth algebraic variety over k.

- The derived moduli stack $\mathrm{Perf}^{\nabla}(X)$ is nil-complete and infinitesimally cartesian.
- $\mathrm{Perf}^{\nabla}(X)$ has a cotangent complex which is perfect at all field valued points.

The formal boundary (i)

Let $\mathfrak{X} \supset X$ be a good compactification: $\mathfrak{X}$ is smooth and proper over k, and $D = \mathfrak{X} - X$ is a simple normal crossings divisor. For an étale map $u : \operatorname{Spec} A \to \mathfrak{X}$ set

I = the ideal of $u^*D \subset \operatorname{Spec} A$;

$\widehat{A} = \lim_n A/I^n$;

$\widehat{\mathfrak{X}}_D$ - the formal completion of $\mathfrak{X}$ along D;

and define derived stacks $\operatorname{Perf}(\widehat{\mathfrak{X}_D})$ and $\operatorname{Perf}(\widehat{\partial}X)$ whose points over a derived affine scheme $S = \operatorname{RSpec}(B)$ are

$$\operatorname{Perf}\left(\widehat{\mathfrak{X}_D}\right)(S) = \lim_{Spec\ A \to \mathfrak{X}} \operatorname{Perf}(\operatorname{Spec} \widehat{A \otimes B}),$$

$$\operatorname{Perf}(\widehat{\partial}X)(S) = \lim_{Spec\ A \to \mathfrak{X}} \operatorname{Perf}(\operatorname{Spec} \widehat{A \otimes B} - V(I)).$$

The formal boundary (ii)

Proposition: **[BeTe],[Ef],[HePoVe]** The k-linear dg category of global points $\mathrm{Perf}(\widehat{\partial} X)(k)$ is independent of the choice of a good compactification $X \subset \mathfrak{X}$.

Note: The proof relies on the rigid tubular descent of **[BeTe]** which only works for smooth varieties. It is unknown if $\mathrm{Perf}(\widehat{\partial} X)(S)$ is independent of $\mathfrak{X}$ for a general affine derived scheme S (even for a singular affine scheme S).

The formal boundary (iii)

Remedy: Work with extendable perfect complexes.
Consider

$$\mathrm{Perf}^{ex}(\widehat{\partial}X) \subset \mathrm{Perf}(\widehat{\partial}X)$$

defined as the Karoubian image of the map of ∞-stacks $\mathrm{Perf}(\widehat{\mathfrak{X}}_D) \to \mathrm{Perf}(\widehat{\partial}X)$.

Proposition: [Efimov,P-Töen]

(a) For any $S \in \mathrm{dAff}_k$ the dg category $\mathrm{Perf}^{ex}(\widehat{\partial}X)(S)$ of extendable perfect complexes is independent of the choice of $X \subset \mathfrak{X}$.

(b) The derived stack $\mathrm{Perf}^{ex}(\widehat{\partial}X)$ is independent of $\mathfrak{X}$.

The formal boundary (iv)

For an étale map $u : \operatorname{Spec} A \to \mathfrak{X}$ and an affine derived scheme $S = \operatorname{RSpec} B$ set

I = the ideal of $u^* D \subset \operatorname{Spec} A$;

$\widehat{\mathrm{DR}}_B(A) = \lim_n \mathrm{DR}(A/I^n \otimes_k B)$ as a B-linear mixed cdga;

$\widehat{\mathrm{DR}}^o_B(A)$ - $\widehat{\mathrm{DR}}_B(A)$ with the local equation of D inverted.

Definition:

(a) $\mathrm{Perf}^\nabla(\widehat{\partial} X)(S)$ is the dg category of sheaves of graded mixed $\widehat{\mathrm{DR}}^o_B(A)$-dg modules which are locally free of weight zero.

(b) The derived pre-stack $\mathrm{Perf}^{\nabla, ex}(\widehat{\partial} X)$ is the fiber product $\mathrm{Perf}^\nabla(\widehat{\partial} X) \times_{\mathrm{Perf}(\widehat{\partial} X)} \mathrm{Perf}^{ex}(\widehat{\partial} X)$.

The formal boundary (v)

Proposition:

(a) The derived pre-stacks $\mathrm{Perf}^{\nabla}(\widehat{\partial}X)$ and $\mathrm{Perf}^{\nabla,ex}(\widehat{\partial}X)$ are stacks.

(b) The derived stack $\mathrm{Perf}^{\nabla,ex}(\widehat{\partial}X)$ is independent of $\mathfrak{X}$.

(c) The restriction map $R : \mathrm{Perf}^{\nabla}(X) \to \mathrm{Perf}^{\nabla}(\widehat{\partial}X)$ is a map of derived stacks which factors through $\mathrm{Perf}^{\nabla,ex}(\widehat{\partial}X)$.

(d) $\mathrm{Perf}^{\nabla}(\widehat{\partial}X)$ is nil-complete, inf-cartesian, and has a cotangent complex which is perfect over all field valued points.

Poisson structures

Theorem:

(i) The morphism $R : \mathrm{Perf}^{\nabla}(X) \to \mathrm{Perf}^{\nabla}(\widehat{\partial} X)$ carries a natural $(2-2n)$-shifted isotropic structure.

(ii) The isotropic structure in (i) is Lagrangian over all field valued points.

Derived moduli of local systems (i)

The derived stack of G local systems can be viewed as an ∞-functor

$$\begin{aligned} Loc_G(X) : \quad \mathrm{cdga}_k^{\leqslant 0} & \longrightarrow \mathrm{SSets} \\ A & \longrightarrow \mathrm{Map}\left(S(X), BG(A)\right) \end{aligned}$$

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singular simplices in X

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simplicial set of A-points of BG

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Note: $Loc_G(X)$ admits a nice quotient presentation.

Derived moduli of local systems (ii)

Choose $\Gamma_\bullet$ - a free simlicial model of the loop group $\Omega_x(X)$ of loops based at $x \in X$.

Derived moduli of local systems (ii)

Choose $\Gamma_\bullet$ - a free simlicial model of the loop group $\Omega_x(X)$ of loops based at $x \in X$.
Note: $B\Gamma_\bullet$ is a simplicial free resolution of the pointed homotpy type (X, x).

Derived moduli of local systems (ii)

Choose $\Gamma_\bullet$ - a free simlicial model of the loop group $\Omega_x(X)$ of loops based at $x \in X$.

Then:

- $R_G(\Gamma_\bullet)$ is a cosimplicial affine k-scheme;
- $\Gamma(R_G(\Gamma_\bullet), \mathcal{O})$ is a commuttative simplicial k-algebra.

Passing to normalized chains gives a $\mathscr{A}_G(X) \in \mathrm{cdga}_k^{\leqslant 0}$ which up to quasi-isomorphism is independent of the choice of the resolution $\Gamma_\bullet$.

Derived moduli of local systems (iii)

The conjugation action of G on $R(\Gamma_\bullet)$ gives an action of G on the cdga $\mathscr{A}_G(X)$ and hence on the derived affine scheme $\mathrm{RSpec}\,\mathscr{A}_G(X)$. The quotient stack

$$Loc_G(X) = [\mathrm{RSpec}\,\mathscr{A}_G(X)/\,G]$$

is the **derived stack of G-local systems on X**.

Back

Orientations and structures (i)

Key observation: Lagrangian structures on a map between moduli of local systems exist always in the presence of relative orientations.

Orientations and structures (i)

$f : Y \to X$ - a continuous map between finite CW complexes; $C^\bullet(Y, X)$ - the cone of the pull-back map $f^* C^\bullet(X) \to C^\bullet(Y)$ on singular cochains with coefficients in k.

An **orientation of dimension d on f** is a morphism of complexes or : $C^\bullet(Y, X) \longrightarrow k[1-d]$, which is non-degenerate in the sense that the pairing

$$C^\bullet(X) \otimes C^\bullet(X, Y) \longrightarrow k[1-d]$$

given by the composition of or with the cup product on $C(X)$ is non-degenerate on cohomology and induces a quasi-isomorphism $C^\bullet(Y, X) \simeq C^\bullet(X)^\vee[1-d]$.

Orientations and structures (ii)

$f : Y \to X$ - continuous map of CW complexes equipped with a relative orientation of dimension d.
G - a reductive algebraic group over k.

Theorem: [Calaque,Brav-Dyckerhoff] The pullback map on the derived stacks of local systems

$$f^* : Loc_G(X) \longrightarrow Loc_G(Y)$$

carries a $(2-d)$-shifted Lagrangian structure which is canonical up to a choice of a non-degenerate element in $\mathrm{Sym}^2(\mathfrak{g}^\vee)^G$.

Back

Poisson bivectors

For a G-local system $\rho \in Loc_G(X)$ we have

- $\mathbb{T}_{Loc_G(X),\rho} = H^\bullet(X, ad(\rho))[1]$
- the bivector p underlying the $(2-d)$-shifted Poisson structure on $Loc_G(X)$ is given by

$$\begin{array}{ccc}
k & \xrightarrow{\quad p \quad} & (H^\bullet(X, ad(\rho))[1] \otimes H^\bullet(X, ad(\rho))[1])[d-2] \\
 & \searrow^{\text{PD}} & \uparrow \\
 & & H^\bullet(X, ad(\rho))[1] \otimes H^\bullet(X, \partial X; ad(\rho))[d-2]
\end{array}$$

Back

Obstructions - smooth D (i)

Caution: The derived stack $Loc_{Z_i,\alpha_i}(D_i)$ may be empty. Indeed:

- $Loc_{Z_i,\alpha_i}(D_i)(k)$ is the groupoid of G-local systems on $\partial_i X$ whose local monodromy around D_i is conjugate to λ_i.
- A $Z_i/Z(Z_i)$-local system on D_i determines a class in $H^2(D_i, Z(Z_i))$, which is the obstruction to lifting it to a Z_i-local system.
- For $Loc_{Z_i,\alpha_i}(D_i)(k)$ to be non-empty one needs to have a $Z_i/Z(Z_i)$-local system on D_i whose obstruction class matches with the image of α_i under the map $H^2(D_i, \mathbb{Z}) \to H^2(D_i, Z(Z_i))$ given by $\lambda_i : \mathbb{Z} \to Z(Z_i)$.

Obstructions - smooth D (ii)

Example: If G is semisimple, k is algebraically closed, and λ_i is a regular semi-simple element, then Z_i is a maximal torus in G and hence the image of α_i in $H^2(D_i, Z_i)$ is zero. If λ_i is of infinte order, this forces α_i to be a torsion class in $H^2(D_i, \mathbb{Z})$.

Back

Obstructions - two components (i)

Definition: A pair of commuting elements $(\lambda_1, \lambda_2) \in G \times G$ is called **strict** if the morphism

$$BZ_{12} \longrightarrow [Z_1/Z_1] \times_{[G*G/G]} [Z_2/Z_2]$$

is Lagrangian (for its canonical isotropic structure).

Here $G * G \subset G \times G$ is the commuting variety, and Z_{12} is the centralizer of the pair (λ_1, λ_2).

Note: Strictness is a group theoretic property.

Obstructions - two components (ii)

Proposition: Let (λ_1, λ_2) be a commuting pair of elements in G, and $u := \mathrm{Id} - \mathrm{ad}(\lambda_1)$ and $v := \mathrm{Id} - \mathrm{ad}(\lambda_2)$ be the corresponding endormorphisms of $\mathfrak{g}$. Then the pair (λ_1, λ_2) is strict if and only u is strict with respect to the kernel of v, i.e. if and only if

$$\mathrm{Im}(v_{|\ker(u)}) = \mathrm{Im}(v) \cap \ker(u).$$

Note: Strictness is symmetric by definition so equivalently (λ_1, λ_2) is strict if and only if v is strict with respect to the kernel of u.

Obstructions - two components (iii)

Corollary:

- If at least one of the λ_i is semi-simple then the pair (λ_1, λ_2) is strict.
- If (u, v) form a principal nilpotent pair [Ginzburg], then the pair (λ_1, λ_2) is strict.

Caution: Strictness is a non-trivial condition: if λ is any non-trivial unipotent element in G, then the pair (λ, λ) is not strict. In this case u is a non-zero nilpotent endomorphism of $\mathfrak{g}$ and thus $\ker(u) \cap \mathrm{Im}(u) \neq 0$, but $\mathrm{Im}(u_{|\ker(u)}) = 0$).

Back

Infinitesimal properties (i)

Note: These are the properties neeeded for applying the Artin-Lurie representability theorem.

Recall that for any $B \in \mathrm{cdga}_k^{\leqslant 0}$, any connective B-module M, and any k-linear derivation $d : B \to M[1]$, the square zero extension $B \oplus_d M$ of B by M is defined by the cartesian square of cdga:

$$\begin{array}{ccc} B \oplus_d M & \longrightarrow & B \\ \downarrow & & \downarrow 0 \\ B & \xrightarrow[d]{} & B \oplus M[1] \end{array}$$

where 0 denotes the natural inclusion of B as a direct factor in the trivial square zero extension $B \oplus M[1]$.

Infinitesimal properties (ii)

Definition: Let F be a derived stack.

- We say that F is **infinitesimally cartesian** if for any B, M and d as above the square

$$\begin{array}{ccc} F(B \oplus_d M) & \longrightarrow & F(B) \\ \downarrow & & \downarrow 0 \\ F(B) & \xrightarrow[d]{} & F(B \oplus M[1]) \end{array}$$

  is cartesian.
- We say that F is *nil-complete* if for any $B \in \mathrm{cdga}_k^{\leqslant 0}$ with Postnikov tower $\{B_{\leqslant n}\}_n$ the natural morphism $F(B) \longrightarrow \lim_n F(B_{\leqslant n})$ is an equivalence.

Back