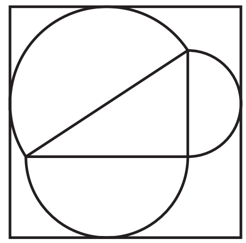
Walker Prize Examination – April 18, 2013

1. Semicircles are constructed on the sides of a right triangle, and then a rectangle is circumscribed about the resulting figure, as shown below. Show that the circumscribed rectangle is a square.



2. Six teams compete in a tournament. Every team plays every other team exactly once. In every game, one of the teams wins (there are no ties). Prove that there is some team x such that for every other team y, either x beats y or there is a third team z such that x beats z and z beats y.

3. Let $\{a_n\}_{n=1}^{\infty}$ be any sequence of positive real numbers, and consider the sequence $\{b_n\}_{n=1}^{\infty}$ defined by
$$b_n = \frac{a_n}{2013 + n^3 a_n^2}.$$
Either prove that the sum $\sum_{n=1}^{\infty} b_n$ always converges, or find a sequence $\{a_n\}_{n=1}^{\infty}$ such that the sum $\sum_{n=1}^{\infty} b_n$ diverges.

4. How many lists $(a_1, a_2, \ldots, a_{100})$ of 100 positive integers are there such that the least common multiple of the numbers in the list is 20? Note that the numbers in the list need not be distinct.

5. You have a collection of blank cards. You want to write three positive integers on each card in such a way that

   - for any two distinct cards, there is exactly one positive integer that is on both of them, and
   - there is no positive integer that is on all of the cards.

   What is the largest number of cards for which this can be done?

   For example, for three cards, you could write:

   card 1: 1, 2, 3

   card 2: 3, 4, 5

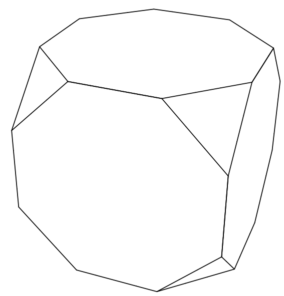
   card 3: 2, 4, 6

6. Let A, B, C, and D be four distinct points lying on a line ℓ. Show that there is some square in the plane whose sides are not parallel to ℓ such that the lines containing the four sides of the square intersect the line ℓ at the four points A, B, C, and D.

7. Let $a_1 = 1$, and for $n > 1$ let $a_n = 2a_{n-1} + (-1)^n$. Find

$$\lim_{n\to\infty} \frac{a_n}{2^{n+1}}.$$

8. Congruent pieces are sliced off each of the eight corners of a cube, resulting in the solid shown below. If the 36 edges of this solid all have length 1, what is the volume of the solid?



9. For positive integers m and n let

$$F(m,n) = m!(n+1)\sum_{i=0}^{m} \frac{(m+n-i)!}{(m-i)!}.$$

Show that $F(m,n) = F(n,m)$. (Recall that $0! = 1$, and for $k > 0$, $k! = 1 \cdot 2 \cdots k$.)

10. Suppose p is a polynomial of degree n with real coefficients and for all real numbers x, $p(x) \geq 0$. Show that for all real numbers x,

$$p(x) + p'(x) + p''(x) + \cdots + p^{(n)}(x) \geq 0.$$