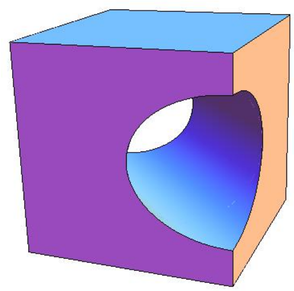
Walker Prize Examination – April 21, 2015

1. Alice and Bob are both walking up a moving escalator. The escalator moves at a constant speed, and Alice and Bob both walk up the steps of the escalator at a constant rate (measured as the number of steps they climb per second), but Alice walks twice as fast as Bob. When she gets to the top, Alice has taken 28 steps. On the other hand, when he gets to the top, Bob has taken only 21 steps. How many steps are visible on the escalator at any given moment?

2. A solid cube is 3 inches on each side. A circular hole with radius 1 inch is drilled through the cube from the center of one edge to the center of the opposite edge (see figure). What is the volume of the remaining solid?



3. Alice and Bob agree to meet at the war memorial at noon. Unfortunately, neither is very punctual. Each of them arrives at a random time between 12:00 and 12:15. If Bob is not there when Alice arrives, she is willing to wait 9 minutes for him; if he hasn't arrived by 9 minutes after she arrives, she will leave. Bob is only willing to wait 7 minutes for Alice. What is the probability that they will meet?

4. Find all solutions to the following system of equations:

$$\begin{aligned} x^2 - yz &= 1 \\ y^2 - xz &= 2 \\ z^2 - xy &= 3 \end{aligned}$$

5. Let S be an infinite set of rectangles in the xy-plane. Each of the rectangles in S has sides parallel to the axes, one corner at $(0,0)$, and one corner at a point with positive integer coordinates. Show that there must exist distinct rectangles A and B in S such that the interior of A is contained in the interior of B.

6. Prove that for every positive integer n, the total number of digits in all the integers from 1 to 10^n is the same as the total number of occurrences of the digit 0 in all the integers from 1 to 10^{n+1}.

7. Find

$$\sum_{n=1}^{\infty} \ln\left(\frac{3^{1/2^n}+1}{2^{1/2^n}+1}\right) = \ln\left(\frac{\sqrt{3}+1}{\sqrt{2}+1}\right) + \ln\left(\frac{\sqrt[4]{3}+1}{\sqrt[4]{2}+1}\right) + \cdots .$$

8. Let $\{a_n\}_{n=1}^{\infty}$ be a strictly increasing arithmetic sequence of positive real numbers, and for each n let

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{a_{n+1} + a_{n+2} + \cdots + a_{2n}}.$$

(a) Find $\lim_{n\to\infty} b_n$.

(b) For which initial sequences $\{a_n\}_{n=1}^{\infty}$ is the sequence $\{b_n\}_{n=1}^{\infty}$ a constant sequence?

(Note: An *arithmetic sequence* is a sequence $\{a_n\}_{n=1}^{\infty}$ with the property that for every n, $a_{n+1} - a_n = a_{n+2} - a_{n+1}$.)

9. Suppose that $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial with real coefficients, and the coefficients satisfy the equation

$$\frac{a_n}{n+1} + \frac{a_{n-1}}{n} + \cdots + \frac{a_1}{2} + a_0 = 0.$$

Prove that $P(c) = 0$ for some real number c between 0 and 1.

10. Show that there are no positive integer solutions to

$$a^4 + b^4 + c^4 + d^4 + e^4 = 16abcde.$$