Integrable Systems on the Dual of Nilpotent Lie Subalgebras and T-Poisson Cluster Structures

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0. Motivation

T: torus $\overset{\text{Pois.}}{\underset{\text{auto.}}{\curvearrowright}}$ (Y, π): Poisson variety.

Def A T-leaf is a subvariety of Y of the form

$$\bigcup_{t \in T} t.S,$$

where $S \subseteq (Y, \pi)$ is a symplectic leaf.

Folklore (Gekhtman-Shapiro-Vainshtein) On the coordinate ring of each T-leaf, there exists a cluster algebra structure which is compatible with the T-action and the Poisson structure.

Compatibility, at least, means that one can find a local coordinate chart $y_1, \cdots, y_n$ (<u>T-chart</u>) on the T-leaf such that:

- Each y_i is a T-eigenfunction;
- $\{y_i, y_j\} = \omega_{ij} y_i y_j$ $\exists \omega_{ij} \in \mathbb{C}$ (log-canonical).

<u>Ex</u> 1. Double Bruhat cell $G^{u,v} := BuB \cap B_- v B_-$.

2. Reduced double Bruhat cell $L^{u,v} := N\bar{u}N \cap B_- v B_-$ $(\cong G^{u,v}/T)$.

3. Bott-Samelson charts.

<u>Rmk</u> Most known examples of T-leaves that admits a compatible cluster algebra structure have to do with (G, π_{st}). The Kirillov-Kostant Poisson structure rarely appears.

$\mathfrak{g}$: semisimple Lie algebra

$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_-$: triangular decomposition

Lie $T = \mathfrak{t}$. T acts on $\mathfrak{n}_-^*$ by the coadjoint action.

<u>Fact</u> (Kostant) The Poisson variety $(\mathfrak{n}_-^*, \pi_{KK})$ contains an open dense T-leaf. Moreover, this open dense T-leaf can be explicitly described using the notion of "cascade of roots".

<u>Natural Question</u> Is there a cluster algebra structure on $\mathbb{C}[\mathfrak{n}_-^*]$ which is compatible with the T-action and the Kirillov-Kostant Poisson structure?

We have constructed an integrable system on $(\mathfrak{n}_-^*, \pi_{KK})$, and the functions in our integrable system are T-eigenfunctions.

Hence, we have found "half of a T-chart".

1. <u>Construction of our integrable system via representation theory</u>

Inspired by a construction due to Kostant - Lipsman - Wolf.

λ: dominant integral weight $\rightsquigarrow$ $V(\lambda)$: highest weight λ representation

$V(\lambda)^*$: dual representation

$w \in W = $ Weyl group $\rightsquigarrow$ $v_{w.\lambda} \in V(\lambda)$: weight vector of weight $w.\lambda$

$z_{-\lambda} \in V(\lambda)^*$: lowest weight vector dual to $v_\lambda \in V(\lambda)$

$\rightsquigarrow f_{w,\lambda} : \mathcal{U}\mathfrak{g} \to \mathbb{C}, \quad x \mapsto \langle z_{-\lambda}, x.v_{w.\lambda} \rangle.$

<u>Def</u> Let $\mathfrak{s}$ be a Lie algebra and $g \in (\mathcal{U}\mathfrak{s})^*$ a linear functional on $\mathcal{U}\mathfrak{s}$. The <u>codegree</u> of g is the minimal integer c such that

$$g|_{\mathcal{U}^{\leq c}\mathfrak{s}} \neq 0.$$

$cd_{w,\lambda} :=$ codegree of $f_{w,\lambda}$.

<u>Def</u> The <u>inversion set</u> $N(w)$ of w consists of those positive roots α such that $w^{-1}.\alpha$ is a negative root.

The Lie subalgebra $\mathfrak{n}_w$ of $\mathfrak{n}$ is defined to be

$$\mathfrak{n}_w := \bigoplus_{\alpha \in N(w)} \mathfrak{g}_\alpha .$$

Consider:

$$\begin{tikzcd}
U^{\leq cd_{w,\lambda}}(\mathfrak{n}_w) \arrow[r, hook] & U(\mathfrak{n}_w) \arrow[r, hook] \arrow[d, phantom, "\circlearrowright"] & U\mathfrak{n} \arrow[r, hook] & U\mathfrak{g} \arrow[r, "f_{w,\lambda}"] & \mathbb{C} . \\
\mathrm{Sym}^{cd_{w,\lambda}}(\mathfrak{n}_w) \arrow[u, "\mathrm{Sym}"] \arrow[rrrru, dashed, "f_{(w,\lambda)}"'] & & & &
\end{tikzcd}$$

Get $f_{(w,\lambda)} \in \mathrm{Sym}^{cd_{w,\lambda}}(\mathfrak{n}_w)^* \cong \mathrm{Sym}^{cd_{w,\lambda}}(\mathfrak{n}_w^*) \cong \{$homogeneous plynomials on $\mathfrak{n}_w$ of degree $cd_{w,\lambda}\} \subseteq \mathbb{C}[\mathfrak{n}_w] \underset{\text{form}}{\overset{\text{Killing}}{\cong}} \mathbb{C}[\mathfrak{n}^*_{w,-}]$.

We apply this construction to the following situation.

$w_0 \in W$: longest element.

$\underline{w_0} = s_{i_1} s_{i_2} \cdots s_{i_N}$: reduced word.

$\forall j \in [1, N]$, put $w_j := s_{i_1} \cdots s_{i_j}$, $cd_j := cd_{w_j, \omega_{i_j}}$.

<u>Def</u> $\forall j \in [1, N]$, define a linear functional

$$f_j := f_{w_j, \omega_{i_j}} : \mathcal{U}\mathfrak{g} \to \mathbb{C}$$

and a polynomial

$$f_{(j)} := f_{(w_j, \omega_{i_j})} \in \mathbb{C}[\mathfrak{n}^*_{w_j,-}].$$

<u>Rmk</u> We can regard $f_{(j)}$ as a polynomial on $\mathfrak{n}^*_-$, homogeneous of degree cd_j.

<u>Thm</u> Any maximal algebraically independent subset of

$$\{f_{(1)}, \cdots, f_{(N)}\}$$

is an integrable system on $(\mathfrak{n}^*_-, \pi_{KK})$.

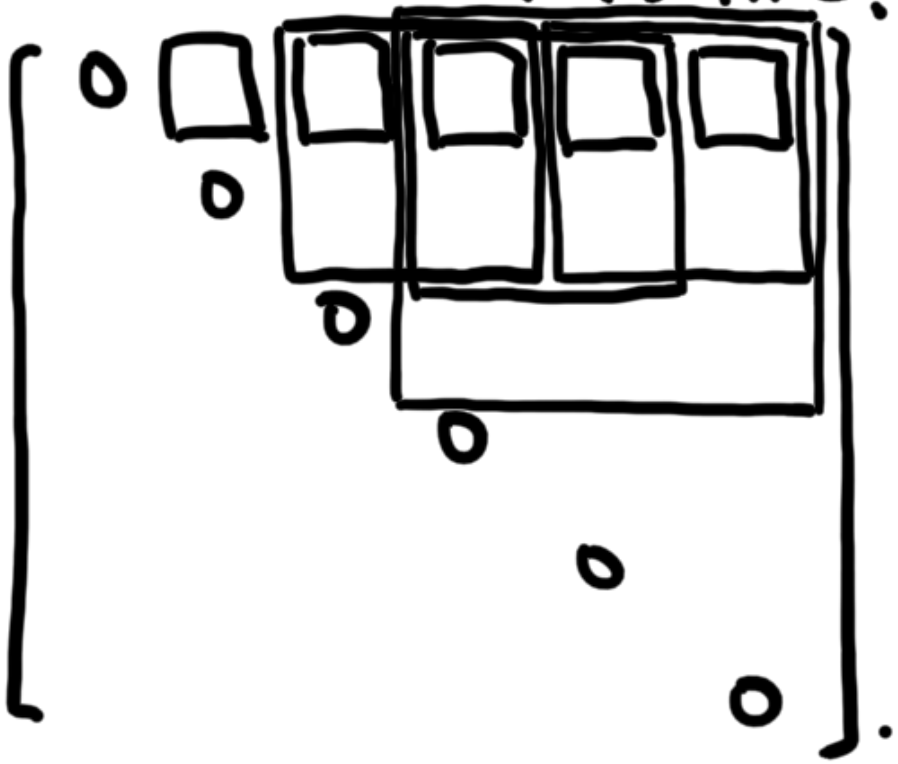
<u>Ex</u> A_5. $\underline{w_0} = s_1 s_2 s_3 s_4 s_5 s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_1 s_2 s_1$.

Compute f_1. $V(\omega_{i_1}) = \mathbb{C}^6 \ni v_{w_1 . \omega_{i_1}} = e_2$

$V(\omega_{i_1})^* = \mathbb{C}^6 \ni z_{-\omega_{i_1}} = e_1$. $\rightsquigarrow f_1 : \mathcal{U}\mathfrak{sl}_6 \to \mathbb{C}$, $x \mapsto \langle e_1, x.e_2 \rangle$.

$\rightsquigarrow cd_1 = 1$ because $f_1(x) = 1$ for $x = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}$. $\rightsquigarrow f_{(1)} = \begin{bmatrix} 0 & \boxed{*} & * & * & * & * \\ & 0 & * & * & * & * \\ & & 0 & * & * & * \\ & & & 0 & * & * \\ & & & & 0 & * \\ & & & & & 0 \end{bmatrix}$

Our integrable system in this case looks like:



2. Cluster algebraic reinterpretation

Bott-Samelson coordinates:

$$\mathbb{C}^N \xrightarrow{\cong} Bw_0B/B, \quad (x_1, \cdots, x_N) \mapsto \exp(x_1 e_{i_1})\bar{s}_{i_1} \cdots \exp(x_N e_{i_N})\bar{s}_{i_N} B/B.$$

Rmk
- $(Bw_0B/B, \pi_{st})$ contains an open dense T-leaf.
- $\mathbb{C}[Bw_0B/B] \cong \mathbb{C}[x_1, \cdots, x_N]$ has the structure of a symmetric Poisson CGL extension, hence has a cluster algebra structure. (Goodearl-Yakimov)

• The <u>homogeneous Poisson primes</u> in $\mathbb{C}[x_1, \cdots, x_N]$ play a crucial role in the construction of the cluster algebra structure.

Up to multiplication by nonzero scalars, the homogeneous Poisson primes are, for $j \in [1, N]$,

$$y_j := \Delta_{\omega_{i_j}, \omega_{i_j}}(\exp(x_1 e_{i_1}) \bar{s}_{i_1} \cdots \exp(x_j e_{i_j}) \bar{s}_{i_j}).$$

<u>Def</u> For $j \in [1, N]$, define y_j^{low} to be the lowest degree terms of the polynomial y_j.

$\lambda_j := s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$: positive root.

$\mathbb{C}^N \xrightarrow{\cong} \mathfrak{n}$, $(x_1, \cdots, x_N) \mapsto x_1 e_{\lambda_1} + \cdots + x_N e_{\lambda_N}$. $\rightsquigarrow \mathbb{C}[\mathfrak{n}_-^*] \underset{\text{form}}{\overset{\text{Killing}}{\cong}} \mathbb{C}[\mathfrak{n}] \cong \mathbb{C}[x_1, \cdots, x_N]$

$\rightsquigarrow f_{(1)}, \cdots, f_{(N)} \in \mathbb{C}[x_1, \cdots, x_N]$.

<u>Thm</u> For $j \in [1, N]$, we have $y_j^{low} = f_{(j)}$.

<u>Why Poisson commute?</u>

$\pi_{st} = \pi_{...} +$ higher order terms

$\pi_{st} = \pi_{KK}$ + higher order terms

Log-canonical: $\{y_i, y_j\}_{st} = \omega_{ij} y_i y_j$.

$\left.\begin{array}{l} \text{Lowest degree term of LHS: } \{y_i^{low}, y_j^{low}\}_{KK} \quad (\text{degree} = \deg y_i^{low} + \deg y_j^{low} - 1) \\ \text{Lowest degree term of RHS: } \omega_{ij} y_i^{low} y_j^{low} \quad (\text{degree} = \deg y_i^{low} + \deg y_j^{low}) \end{array}\right\}$

$\rightsquigarrow \{y_i^{low}, y_j^{low}\}_{KK} = 0$.

3. Generalization

A similar construction gives rise to an integrable system on $(\mathcal{O}^*_{w,-}, \pi_{KK})$.

The T-Poisson Pfaffian Pf_w of $(BwB/B, \pi_{st})$ plays a key role in this construction.

By definition, Pf_w is a section of the top exterior power of the tangent bundle of BwB/B, so $Pf_w = f_w \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_{l(w)}}$ for some polynomial f_w in the Bott-Samelson coordinates $x_1, \cdots, x_{l(w)}$.

Let f_w^{low} be the lowest degree term of f_w and d_w the degree of f_w^{low}.

A consequence of our construction is:

<u>Cor</u> Let $2r_w$ be the maximum of the dimension of symplectic leaves in $(\mathfrak{n}^*_{w,-}, \pi_{KK})$. Then we have

$$r_w = l(w) - d_w.$$

In other words, the <u>index</u> of the Lie algebra $\mathfrak{n}_w$ (equivalently, $\mathfrak{n}_{w,-}$) is $2d_w - l(w)$.