Symplectic double groupoids in generalized Kähler geometry

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Joint work with Daniel Álvarez and Marco Gualtieri

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Definition

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- The endomorphism $\mathbb{G}$ is called a **generalized metric** which is equivalent to a **Riemannian metric** g on M and an isotropic splitting of E.
- First example of GK structures: Kähler structures, where $\mathbb{J}_1, \mathbb{J}_2$ are given by the symplectic and complex structures respectively and $\mathbb{G}$ is given by the Riemannian metric.

- It turns out that there is rich geometry hidden in the simple definition. First of all, there are holomorphic structures.
- We have a decomposition of the Courant algebrid as eigenspaces of $\mathbb{J}_1$ and $\mathbb{J}_2$:
$$E \otimes \mathbb{C} = \ell_+ \oplus \bar{\ell}_+ \oplus \ell_- \oplus \bar{\ell}_-,$$
where $\ell_+ = L_1 \cap L_2$ and $\ell_- = L_1 \cap \bar{L}_2$. where L_1 and L_2 are the $+i$ eigenbundles of $\mathbb{J}_1$ and $\mathbb{J}_2$ respectively.
Then $\rho(\ell_+)$ and $\rho(\ell_-)$ define complex structures $I_\pm$ such that $\rho(\ell_\pm) = T_\pm^{10}$.
- Moreover, we may reduce E with respect to $\bar{\ell}_\pm$ and obtain holomorphic Courant algebroids $\mathcal{E}_\pm$ via Courant reduction. As bundles, we simply take
$$\mathcal{E}_\pm = \bar{\ell}_\pm^\perp / \bar{\ell}_\pm \cong \ell_\mp \oplus \bar{\ell}_\mp.$$
- We may reduce the complex Dirac structures of $\mathbb{J}_1$ and $\mathbb{J}_2$ as well and obtain
$$\mathcal{A}_+ = (\bar{L}_1)_{\bar{\ell}_+}, \quad \mathcal{A}_- = (\bar{L}_1)_{\bar{\ell}_-}, \quad \mathcal{B}_+ = (\bar{L}_2)_{\bar{\ell}_+}, \quad \mathcal{B}_- = (L_2)_{\bar{\ell}_-}$$
$$\mathcal{A}_+ \cong \bar{\ell}_-, \quad \mathcal{B}_+ \cong \ell_-, \quad \mathcal{A}_- \cong \bar{\ell}_+, \qquad \mathcal{B}_- \cong \ell_+.$$

- In summary, we reduce the generalized complex structures, which a priori are only smooth and obtain **holomorphic Manin triples**, i.e. complementary Dirac structures:

$$\mathcal{A}_\pm \oplus \mathcal{B}_\pm = \mathcal{E}_\pm .$$

- As the reduction are carried out separately, we also need to understand the connections between the $+$ side and the $-$ side.
  We may use the matched pair constructions, and recover the GC structures. For example, consider

$$\mathcal{A}_+ \oplus T_+^{01}, \quad \mathcal{A}_- \oplus T_-^{01}$$

  which are both isomorphic to $\bar{L}_1$. But in order to phrase the compatibility condition, we need a more accurate result.

Theorem

The matched pairs of $\mathcal{A}_+$ and $\mathcal{A}_-$ are gauge equivalent and the matched pairs of $\mathcal{B}_+$ and $\overline{B}_-$ are gauge equivalent.

$$L_{\mathcal{A}_+} = e^{-iF_1} L_{\mathcal{A}_-}, \quad L_{\mathcal{B}_+} = e^{-iF_2} L_{\overline{\mathcal{B}}_-},$$

where F_1, F_2 are real 2-forms which are not necessarily closed.
Conversely, if $(\mathcal{A}_\pm, \mathcal{B}_\pm)$ are holomorphic Manin triples satisfying the above gauge equivalences then they define a GK structure (with a possibly degenerate metric).

- The gauge equivalences

$$L_{\mathcal{A}_+} = e^{-iF_1} L_{\mathcal{A}_-}, \quad L_{\mathcal{B}_+} = e^{-iF_2} L_{\overline{\mathcal{B}}_-}$$

  are difficult to study in general. However, a simpler case, the so called **symplectic type** has been studied extensively in the literature.

- We assume that one of the GC structures is of symplectic type, i.e. gauge equivalent to a symplectic structure. Then the holomorphic Manin triples simplify to

$$\mathrm{Gr}((2i)^{-1}\sigma_\pm) \oplus T^{10}_\pm = \mathcal{E}_\pm,$$

  where $\sigma_\pm$ are holomorphic Poisson structures with respect to $I_\pm$. The gauge equivalences then simplify to

$$L_{\sigma_+} = e^F L_{\sigma_-}.$$

  The equation can be written in terms of tensors as follows.

$$I_+ - I_- = QF, \quad I_-^* F + FI_+ = 0,$$

  where $Q = -4\mathrm{Im}(\sigma_+) = -4\mathrm{Im}(\sigma_-)$.

- The above equation can be simplified into a quadratic equation for F:

$$I_-^* F + FI_- + FQF = 0.$$

  In general, this is still a difficult equation. But what if Q is invertible?

- Assume that the holomorphic Poisson structures σ_+ and σ_- are nondegenerate. Let $\omega_\pm = -\sigma_\pm^{-1}$. Then the gauge equivalence reduces to a simple-looking equation:

$$\omega_+ - \omega_- = F.$$

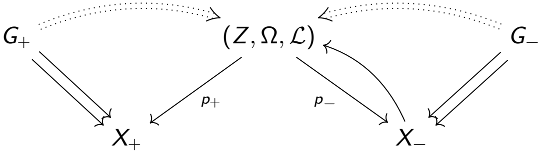
- To implement the above simplifications, the main idea will be to transgress the structures via integration of infinitesimal structures, e.g. integration of Lie algebroids into Lie groupoids.
- The gauge equation $\omega_+ - \omega_- = F$ will then be understood as Morita equivalence between the global structures. In order to encode F, a bisection will be needed.

- An application of this idea is the following theorem of F. Bischoff, M. Guatieri, and M. Zabzine.

Theorem (F. Bischoff, M.Gualtieri, M. Zabzine)

A symplectic type GK is equivalent to a holomorphic symplectic Morita equivalence with an imaginary Lagrangian bisection between symplectic groupoids.

We remark that the holomorphic symplectic Morita equivalence induces the underlying holomorphic data of the GK structure and the Im-Lagrangian bisection, which is a smooth data induces the other underlying smooth data of the GK structure, e.g. the metric g.



- The holomorphic Poisson structures $\sigma_\pm$ are induced from $G_\pm$. The gauge F is the pullback of Ω to $\mathcal{L}$.

- As an example, let (M, F, I) be a Kähler structure. Then $\sigma = 0$. The holomorphic symplectic groupoid G_- is $(T^*M, \omega_0) \rightrightarrows M$. The Morita equivalence is
$$Z = (T^*M, \omega_0 + \pi^* F).$$
The bisection $\mathcal{L}$ is the zero section of T^*M and induces the Kähler form F simply by pullback of $\omega_0 + \pi^* F$ to $\mathcal{L}$.

- To deal with the general case, we inevitably need to consider integration of **Lie bialgebroids**. A general result in Poisson geometry due to K. Mackenzie, A. Weinstein, P. Xu is that a Lie bialgebroid should be integrated into a **symplectic double groupoids** D whose side groupoids G_A and G_B are Poisson groupoids in duality and induce the underlying Lie bialgebroids:

$$\begin{array}{ccc} D & \rightrightarrows & G_A \\ \downdownarrows & & \downdownarrows \\ G_B & \rightrightarrows & X \end{array}$$

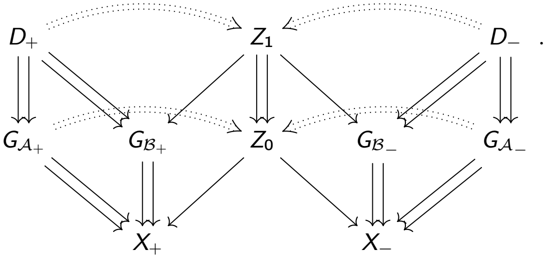
- So we let $G_{\mathcal{A}_\pm}$ and $G_{\mathcal{B}_\pm}$ be the holomorphic Poisson groupoids integrating the Lie bialgebroids $(\mathcal{A}_\pm, \mathcal{B}_\pm)$ and $(\mathcal{B}_\pm, \mathcal{A}_\pm)$ respectively.
- The gauge equivalences

$$L_{\mathcal{A}_+} = e^{-iF_1} L_{\mathcal{A}_-}, \quad L_{\mathcal{B}_+} = e^{-iF_2} L_{\overline{\mathcal{B}}_-}$$

then transgress to symplectic type GK structures on the Poisson groupoids $G_{\mathcal{A}_\pm}$ and $G_{\mathcal{B}_\pm}$. Then we can apply the above theorem.

Symplectic double groupoids

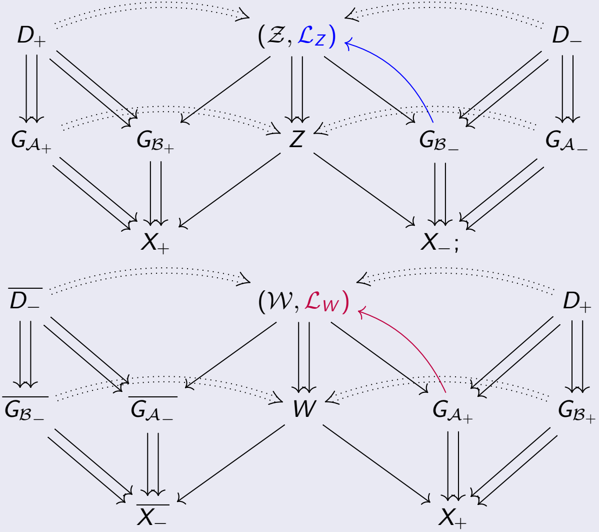
- For example, taking care of the first gauge equivalence, we obtain the following holomorphic symplectic Morita equivalence for symplectic double groupoids:



We remark that here the action of the double groupoid is horizontal meaning that $D_\pm \rightrightarrows G_{\mathcal{B}_\pm}$ act on Z_1 and $G_{\mathcal{A}_\pm}$ act on Z_0.

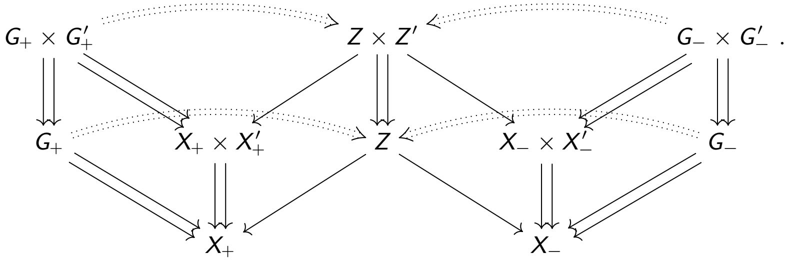
Theorem (Alvarez, Gualtieri, J.)

A GK structure is equivalent to a pair of holomorphic symplectic Morita equivalences between symplectic double groupoids D_+, D_- and $D_+, \overline{D}_-$ with related multiplicative Re-symplectic and Im-Lagrangian bisections $\mathcal{L}_Z$ and $\mathcal{L}_W$.



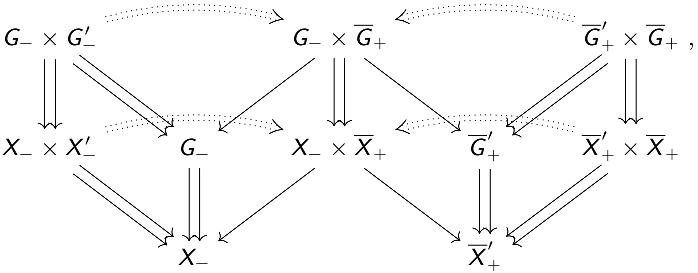
Revisit the symplectic type

- Let us return to the symplectic type and see how the above theorem fits into the previous theorem. Half of the diagram is



where X' means taking the opposite Poisson structure. We see that the base Morita equivalence with Lagrangian bisection is the one in the previous theorem and the upper one is $(Z \times Z, \Omega \times -\Omega)$ induced by doubling. The pullback of $\Omega \times -\Omega$ to the bisection is the multiplicative symplectic form $F \times -F$ on $X \times X \rightrightarrows X$ which differentiates to $\pi_1 = F^{-1}$, which in this particular case, can be used to recover the GK structure.

- We may also consider the other half of the diagram.



where $\overline{X}$ means taking the conjugate complex structure.
The holomorphic Morita equivalence between $(\overline{G}'_+, -\overline{\Omega}_+)$ and (G_-, Ω_-) is simply the product $(G_- \times \overline{G}_+, \Omega_- \times \overline{\Omega}_+)$. The Lagrangian bisection is the diagonal embedding of the underlying smooth groupoid of $G_\pm$. The pullback of $\Omega_- \times \overline{\Omega}_+$ to the bisection is the symplectic form $2\mathrm{Re}\Omega_- + \delta^* F$ which differentiates to π_2, which is not enough to recover the GK structure.

- We remark that from this example, we see that in general one Lagrangian bisection is not enough to recover the entire GK structure.