Solutions to the Algebra problems on the Comprehensive Examination of January 30, 2009

1. **(25 Points).** Let G be a group, and let $H \subseteq G$ be a subgroup. Suppose that for every $x, y \in G$ satisfying $xy \in H$, we have $yx \in H$. Prove that H is a **normal** subgroup of G.

   Solution: Given $g \in G$, $h \in H$, $g^{-1}(gh) = (g^{-1}g)h = h \in H$. Thus by the given property of H, $ghg^{-1} = (gh)g^{-1} \in H$ so H is normal as desired. QED

2. **(25 points).** Recall that S_7 denotes the group of permutations of the set $\{1, 2, \ldots, 7\}$. Define the permutations $\sigma, \tau \in S_7$ by

$$\sigma = (1\ 2)(4\ 6\ 7\ 5) \quad \text{and} \quad \tau = (1\ 3\ 5)(2\ 7\ 4\ 6).$$

   (a) **(10 points).** Write $\sigma\tau$ as a product of disjoint cycles in S_7.

   Solution: $\sigma\tau = (1\ 3\ 4\ 7\ 6)(2\ 5)$

   (b) **(10 points).** Find the order of each of σ, τ, and $\sigma\tau$.

   Solution: The order of σ is the lcm of the orders of each individual (disjoint) cycle (and the order of an n-cycle is n): $o(\sigma) = \text{lcm}(2, 4) = 4$. Similarly, $o(\tau) = \text{lcm}(3, 4) = 12$ and $o(\sigma\tau) = \text{lcm}(2, 5) = 10$.

   (c) **(5 points).** Determine whether each of σ, τ, and $\sigma\tau$ is even or odd.

   Solution: $\sigma = (1\ 2)(4\ 6)(6\ 7)(7\ 5)$ which is the product of 4 transpositions so σ is even. $\tau = (1\ 3)(3\ 5)(2\ 7)(7\ 4)(4\ 6)$ which is the product of 5 transpositions so τ is even. $\sigma\tau$ is therefore the product of $4 + 5 = 9$ transpositions so it is odd.

3. **(25 points).** Let R be a ring.

   (a) **(10 points).** Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.
   If you use other terms like "closed" or "coset" or "subgroup" or "subring" or "maximal" in your definition, you must define those terms as well.

   Solution: $I \subseteq R$ is an ideal of R if $(I, +)$ is a subgroup of $(R, +)$ and $\forall x \in I,\ r \in R$, $xr \in I$ and $rx \in I$. If G is a group, a subgroup is a set $H \subseteq G$ such that H forms a group under G's group operation.

   (b) **(15 points).** Let $I \subseteq R$ be an ideal of R, and suppose that $xy - yx \in I$ for every $x, y \in R$. Prove that the quotient ring R/I is commutative.

   Solution: Given $I + a, I + b \in R/I$, by the given property of I $ab - ba \in H$. Thus $(I + a)(I + b) = I + ab = I + ba = (I + b)(I + a)$, so R/I is commutative as desired. QED

4. **(25 points).** Let $\mathbb{F}_2 = \{0, 1\}$ denote the field of two elements, and let $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ denote the field of five elements.

   (a) **(15 points).** Prove that $f(X) = X^4 + X^2 + 1$ is **reducible** in the polynomial ring $\mathbb{F}_2[X]$.

   Solution: To be reducible, f must either have a degree 1 factor (and hence have a root in $\mathbb{F}_2$), or f must be a product of two irreducible degree-2 polynomials. But $f(0) = f(1) = 1$ so f has no roots, which means that f is the product of two irreducible degree-2 factors. The possible degree-2 polynomials are X^2, $X^2 + 1$, $X^2 + X$ and $X^2 + X + 1$. Over $\mathbb{F}_2$ we have $X^2 + 1 = (X + 1)^2$, so $X^2 + X + 1$ is the only degree-2 irreducible polynomial. Then an explicit calculation (made easier by using $(a + b)^2 = a^2 + b^2$) shows that $f(X) = (X^2 + X + 1)(X^2 + X + 1)$.

   (b) **(10 points).** Prove that $g(X) = X^3 + 2X^2 + 2X + 3$ is **irreducible** in the polynomial ring $\mathbb{F}_5[X]$.

   Solution: Since $\deg g = 3$, if g is reducible, then it must have a degree 1 factor and hence a root in $\mathbb{F}_5$. There are only 5 elements of $\mathbb{F}_5$, so let's check them (remembering that we are working modulo 5):

$$
\begin{aligned}
g(0) &= 0^3 + 2(0^2) + 2(0) + 3 = 0 + 0 + 0 + 3 = 3 \neq 0 \\
g(1) &= 1^3 + 2(1^2) + 2(1) + 3 = 1 + 2 + 2 + 3 = 3 \neq 0 \\
g(2) &= 2^3 + 2(2^2) + 2(2) + 3 = 3 + 3 + 4 + 3 = 3 \neq 0 \\
g(3) &= 3^3 + 2(3^2) + 2(3) + 3 = 2 + 3 + 1 + 3 = 4 \neq 0 \\
g(4) &= 4^3 + 2(4^2) + 2(4) + 3 = 4 + 2 + 3 + 3 = 2 \neq 0
\end{aligned}
$$

   Thus g has no roots in $\mathbb{F}_5$, it is irreducible. QED