Walker Prize Examination 2018
Answers

1. *The Amherst and Williams shuffleboard teams are going to play five matches to decide their championship, with the champion being the team that wins at least three matches. In each match, Amherst independently has a $\frac{2}{3}$ chance of winning. What is the probability that Amherst wins the championship?*

   Answer: $\dfrac{64}{81}$

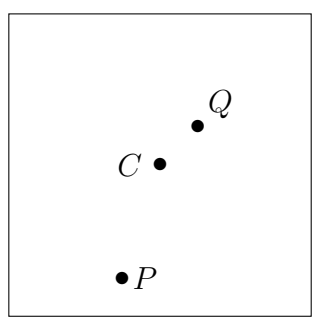
2. *Calculate the integral*
$$\int_0^\infty \frac{1}{1+x+x^2+x^3}\,dx.$$

   Answer: $\dfrac{\pi}{4}$

3. *Find the smallest positive integer n such that the real numbers*
$$\sqrt{n}, \sqrt{n+1}, \ldots, \sqrt{n+2018}$$
   are all equal when rounded to the nearest integer.

   Answer: 1019091

4. *Consider a square with edge length 1 and center C. Let A be the set of points P such that the distance from P to C is less than or equal to the distances from P to each edge of the square. (For example, the picture below shows that the point P is not in the set A, but Q is.) Calculate the area of the set A.*



   Answer: $\dfrac{4\sqrt{2}-5}{3} \approx 0.22$

5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is continuous and*
$$\lim_{x\to\infty} f'(x) = 1.$$
   Show that the limit $\lim_{x\to\infty} f(x)$ diverges.

   (Note: a limit diverges *if either it does not exist or it is equal to $\pm\infty$.)*

6. *Angela and Ben have a total of* 2018 *marbles between them. On January 1, Angela gives Ben the number of marbles equal to how many Ben already has. On January 2, Ben gives Angela the number of marbles equal to how many Angela already has. They continue giving each other marbles, alternating in this way, until the day that one of them does not have enough marbles to give the right number to the other. What is the latest date in the year that day could be?*

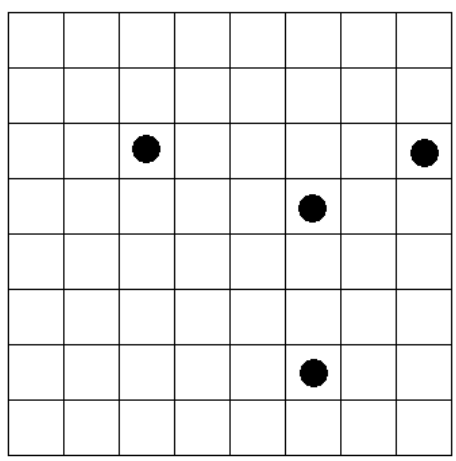
   Answer: January 11

7. *Your First-Year Seminar has 15 students, and you all need to agree on five students from the class to give a presentation. Any two students in the class either are friends with each other or else are both friends with a third student in the class. Prove that it is possible to choose a group of five students such that every student in the class either is in the group or is friends with at least one student in the group.*

8. *Let S be the sum of all fractions of the form $\frac{1}{n}$ where n is a positive integer with exactly two nonzero digits. Is S bigger than 4?*

   Answer: No

9. *What is the largest number of coins that can be placed at the centers of the spaces of a regular 8×8 grid so that no three coins are the vertices of a right-angled triangle? (For example, the diagram below shows that it is possible to place four coins in this way.)*



   Answer: 14 coins

10. *A tetrahedron has vertices $(0,0,0)$, $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ where $a,b,c > 0$. What is the radius of the largest sphere that can be fit inside the tetrahedron? (The sphere may touch, but not cross, the faces of the tetrahedron.)*

    Answer: the radius is $\dfrac{1}{a^{-1}+b^{-1}+c^{-1}+\sqrt{a^{-2}+b^{-2}+c^{-2}}}$